Deep learning

3.2. Probabilistic view of a linear classifier

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Consider the following class populations

\[ \forall y \in \{0, 1\}, x \in \mathbb{R}^D, \]

\[ \mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D|\Sigma|}} \exp \left( -\frac{1}{2} (x - m_y)\Sigma^{-1}(x - m_y)^T \right). \]

That is, they are Gaussian with the same covariance matrix \( \Sigma \). This is the homoscedasticity assumption.
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**homoscedasticity** 

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Intuitively we can map data linearly to make all the covariance matrices identity, there the Bayesian separation is a plan, so it is also in the original space.
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\[ P(Y = 1 \mid X = x) = \sigma \left( \log \frac{\mu_{X \mid Y=1}(x)}{\mu_{X \mid Y=0}(x)} + \log \frac{P(Y = 1)}{P(Y = 0)} \right). \]
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So with our Gaussians $\mu_{X|Y=y}$ of same $\Sigma$, we get

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$$= \sigma \left( \underbrace{(m_1 - m_0)\Sigma^{-1}}_{w} x^T + \underbrace{\frac{1}{2} \left( m_0\Sigma^{-1}m_0^T - m_1\Sigma^{-1}m_1^T \right)}_{b} + Z \right)$$

$$= \sigma (w \cdot x + b).$$

The homoscedasticity makes the second-order terms vanish.
\begin{align*}
\mu_{X|Y=0} & \\
\mu_{X|Y=1} & \\
P(Y = 1 | X = x) & 
\end{align*}
\( \mu_{X|Y=0} \)

\( \mu_{X|Y=1} \)

\( P(Y = 1 | X = x) \)
\[ \mu_{X \mid Y=0} \]

\[ \mu_{X \mid Y=1} \]

\[ P(Y = 1 \mid X = x) \]
$$\mu_X|Y=0$$

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$$P(Y = 1 | X = x)$$
Note that the (logistic) sigmoid function

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So the overall model

\[ f(x; w, b) = \sigma(w \cdot x + b) \]

looks very similar to the perceptron.
We can use the model from LDA

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but instead of modeling the densities and derive the values of \( w \) and \( b \), directly compute them by maximizing their probability given the training data.
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$$f(x; w, b) = \sigma(w \cdot x + b)$$

but instead of modeling the densities and derive the values of $w$ and $b$, directly compute them by maximizing their probability given the training data.

First, to simplify the next slide, note that we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x),$$

hence if $Y$ takes value in $\{-1, 1\}$ then

$$\forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)).$$
We have

\[ \log \mu_{W,B}(w, b \mid \mathcal{D} = d) \]

\[ = \log \frac{\mu_{\mathcal{D}}(d \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(d)} \]

\[ = \log \mu_{\mathcal{D}}(d \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z \]

\[ = \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z' \]
We have

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\log \mu_{W,B}(w, b \mid D = d) = \log \frac{\mu_D(d \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_D(d)} = \log \mu_D(d \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z = \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'
\]

This is the **logistic regression**, whose loss aims at minimizing

\[
-\log \sigma(y_n f(x_n)).
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Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.
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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.
The end