Many applications such as image synthesis, denoising, super-resolution, speech synthesis, compression, etc. require to go beyond classification and regression, and model explicitly a high dimension signal.

This modeling consists of finding “meaningful degrees of freedom” that describe the signal, and are of lesser dimension.
When dealing with real-world signals, this objective involves the same theoretical and practical issues as for classification or regression: defining the right class of high-dimension models, and optimizing them.

Regarding synthesis, we saw that deep feed-forward architectures exhibit good generative properties, which motivates their use explicitly for that purpose.
An autoencoder maps a space to itself and is [close to] the identity on the data. Dimension reduction can be achieved with an autoencoder composed of an encoder $f$ from the original space $\mathcal{X}$ to a latent space $\mathcal{F}$, and a decoder $g$ to map back to $\mathcal{X}$ (Bourlard and Kamp, 1988; Hinton and Zemel, 1994).

If the latent space is of lower dimension, the autoencoder has to capture a "good" parametrization, and in particular dependencies between components.

Let $q$ be the data distribution over $\mathcal{X}$. A good autoencoder could be characterized with the quadratic loss

$$
E_{X \sim q} \left[ \| X - g \circ f(X) \|^2 \right] \simeq 0.
$$

Given two parametrized mappings $f(\cdot; w_f)$ and $g(\cdot; w_g)$, training consists of minimizing an empirical estimate of that loss

$$
\hat{w}_f, \hat{w}_g = \arg\min_{w_f, w_g} \frac{1}{N} \sum_{n=1}^{N} \| x_n - g(f(x_n; w_f); w_g) \|^2.
$$

A simple example of such an autoencoder would be with both $f$ and $g$ linear, in which case the optimal solution is given by PCA. Better results can be achieved with more sophisticated classes of mappings, in particular deep architectures.
A deep autoencoder combines an encoder composed of convolutional layers, with a decoder composed of transposed convolution or other interpolating layers. 

_E.g._ for MNIST:

```python
AutoEncoder (
    (encoder): Sequential (
        (0): Conv2d(1, 32, kernel_size=(5, 5), stride=(1, 1))
        (1): ReLU (inplace)
        (2): Conv2d(32, 32, kernel_size=(5, 5), stride=(1, 1))
        (3): ReLU (inplace)
        (4): Conv2d(32, 32, kernel_size=(4, 4), stride=(2, 2))
        (5): ReLU (inplace)
        (6): Conv2d(32, 32, kernel_size=(3, 3), stride=(2, 2))
        (7): ReLU (inplace)
        (8): Conv2d(32, 8, kernel_size=(4, 4), stride=(1, 1))
    )

    (decoder): Sequential (
        (0): ConvTranspose2d(8, 32, kernel_size=(4, 4), stride=(1, 1))
        (1): ReLU (inplace)
        (2): ConvTranspose2d(32, 32, kernel_size=(3, 3), stride=(2, 2))
        (3): ReLU (inplace)
        (4): ConvTranspose2d(32, 32, kernel_size=(4, 4), stride=(2, 2))
        (5): ReLU (inplace)
        (6): ConvTranspose2d(32, 32, kernel_size=(5, 5), stride=(1, 1))
        (7): ReLU (inplace)
        (8): ConvTranspose2d(32, 1, kernel_size=(5, 5), stride=(1, 1))
    )
```

Deep Autoencoders
**Encoder**

Tensor sizes / operations

\[
\begin{align*}
1 \times 28 \times 28 \\
n.\text{Conv2d}(1, 32, \text{kernel\_size}=5, \text{stride}=1) & \quad 32 \times 24 \times 24 \\
n.\text{Conv2d}(32, 32, \text{kernel\_size}=5, \text{stride}=1) & \quad 32 \times 20 \times 20 \\
n.\text{Conv2d}(32, 32, \text{kernel\_size}=4, \text{stride}=2) & \quad 32 \times 9 \times 9 \\
n.\text{Conv2d}(32, 32, \text{kernel\_size}=3, \text{stride}=2) & \quad 32 \times 4 \times 4 \\
n.\text{Conv2d}(32, 8, \text{kernel\_size}=4, \text{stride}=1) & \quad 8 \times 1 \times 1 \\
\end{align*}
\]

**Decoder**

Tensor sizes / operations

\[
\begin{align*}
8 \times 1 \times 1 \\
n.\text{ConvTranspose2d}(8, 32, \text{kernel\_size}=4, \text{stride}=1) & \quad 32 \times 4 \times 4 \\
n.\text{ConvTranspose2d}(32, 32, \text{kernel\_size}=3, \text{stride}=2) & \quad 32 \times 9 \times 9 \\
n.\text{ConvTranspose2d}(32, 32, \text{kernel\_size}=4, \text{stride}=2) & \quad 32 \times 20 \times 20 \\
n.\text{ConvTranspose2d}(32, 32, \text{kernel\_size}=5, \text{stride}=1) & \quad 32 \times 24 \times 24 \\
n.\text{ConvTranspose2d}(32, 1, \text{kernel\_size}=5, \text{stride}=1) & \quad 1 \times 28 \times 28 \\
\end{align*}
\]
Training is achieved with quadratic loss and Adam

```python
model = AutoEncoder(nb_channels, embedding_dim)
optimizer = optim.Adam(model.parameters(), lr = 1e-3)

for epoch in range(args.nb_epochs):
    for input in train_input.split(batch_size):
        z = model.encode(input)
        output = model.decode(z)
        loss = 0.5 * (output - input).pow(2).sum() / input.size(0)

        optimizer.zero_grad()
        loss.backward()
        optimizer.step()
```

\[ X \text{ (original samples)} \]

\[ g \circ f(X) \text{ (CNN, } d = 8) \]

\[ g \circ f(X) \text{ (PCA, } d = 8) \]
To get an intuition of the latent representation, we can pick two samples \( x \) and \( x' \) at random and interpolate samples along the line in the latent space

\[
\forall x, x' \in \mathcal{X}^2, \; \alpha \in [0, 1], \; \xi(x, x', \alpha) = g((1 - \alpha)f(x) + \alpha f(x')).
\]
And we can assess the generative capabilities of the decoder $g$ by introducing a [simple] density model $q^Z$ over the latent space $\mathcal{F}$, sample there, and map the samples into the image space $\mathcal{X}$ with $g$.

We can for instance use a Gaussian model with diagonal covariance matrix.

$$f(X) \sim \mathcal{N}(\hat{m}, \hat{\Delta})$$

where $\hat{m}$ is a vector and $\hat{\Delta}$ a diagonal matrix, both estimated on training data.
These results are unsatisfying, because the density model used on the latent space $\mathcal{F}$ is too simple and inadequate.

Building a "good" model amounts to our original problem of modeling an empirical distribution, although it may now be in a lower dimension space.
References
