Deep learning

5.4. $L_2$ and $L_1$ penalties

François Fleuret
https://fleuret.org/ee559/
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We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

\[
\log \mu_W(w \mid \mathcal{D} = d) = \log \mu_{\mathcal{D}}(d \mid W = w) + \log \mu_W(w) - \log Z.
\]

If \( \mu_W \) is a Gaussian density with a covariance matrix proportional to the identity, the log-prior \( \log \mu_W(w) \) results in a quadratic penalty

\[
\lambda \|w\|_2^2.
\]

Since this penalty is convex, its sum with a convex functional is convex.

This is called the \( L_2 \) regularization, or “weight decay” in the artificial neural network community.
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

Since the derivative of $\|x\|_2^2$ is zero at zero, the optimal will never move there if it was not already there.
Convnet trained on MNIST with 1,000 samples and a $L_2$ penalty.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Train</th>
<th>Test</th>
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<tbody>
<tr>
<td>0.000</td>
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<td>0.075</td>
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<td>0.020</td>
<td>0.048</td>
<td>0.101</td>
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```python
output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])
for p in model.parameters():
    loss += lambda_l2 * p.pow(2).sum()
optimizer.zero_grad()
loss.backward()
optimizer.step()
```
We can apply the exact same scheme with a Laplace prior

\[
\mu(w) = \frac{1}{(2b)^D} \exp \left( -\frac{\|w\|_1}{b} \right)
\]

\[
= \frac{1}{(2b)^D} \exp \left( -\frac{1}{b} \sum_{d=1}^{D} |w_d| \right),
\]

which results in a penalty term of the form

\[\lambda \|w\|_1.\]

This is the \(L_1\) regularization. As for the \(L_2\), this penalty is convex, and its sum with a convex functional is convex.
An important property of the $L_1$ penalty is that, if $\mathcal{L}$ is convex, and

$$w^* = \arg\min_w \mathcal{L}(w) + \lambda \|w\|_1$$

then

$$\forall d, \left| \frac{\partial \mathcal{L}}{\partial w_d}(w^*) \right| < \lambda \Rightarrow w^*_d = 0.$$
In practice it means that this penalty pushes some of the variables to zero, but contrary to the $L_2$ penalty they actually move and remain there.

The $\lambda$ parameter controls the sparsity of the solution.
With the $L_1$ penalty, the update rule becomes

$$w_{t+1} = w_t - \eta (g_t + \lambda \text{sign}(w_t)),$$

where sign is applied per-component. This is almost identical to

$$w'_t = w_t - \eta g_t$$
$$w_{t+1} = w'_t - \eta \lambda \text{sign}(w'_t).$$

This update may overshoot, and result in a component of $w'_t$ strictly on one side of 0, while the same component in $w_{t+1}$ is strictly on the other.

While this is not a problem in principle, since $w_t$ will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).
The **proximal operator** prevents parameters from “crossing zero”, by adapting \( \lambda \) when it is too large

\[
\begin{align*}
    w'_t &= w_t - \eta g_t \\
    w_{t+1} &= w'_t - \eta \min(\lambda, |w'_t|) \odot \text{sign}(w'_t).
\end{align*}
\]

where \( \min \) is component-wise, and \( \odot \) is the Hadamard component-wise product.
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3$$

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + \frac{1}{2}|x|$$

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + \frac{3}{2}|x|$$

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + |x|$$

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 2|x|$$
Convnet trained on MNIST with 1,000 samples and a $L_1$ penalty.

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<tr>
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<td>0.516</td>
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output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])
optimizer.zero_grad()
loss.backward()
optimizer.step()
with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_1))
```

François Fleuret
Deep learning / 5.4. $L_2$ and $L_1$ penalties
Penalties on the weights may be useful when dealing with small models and small data-sets and are still standard when data is scarce.

While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines.