Deep learning

3.4. Multi-Layer Perceptrons

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A linear classifier of the form

\[
\mathbb{R}^D \rightarrow \mathbb{R} \\
\quad x \mapsto \sigma(w \cdot x + b),
\]

with \( w \in \mathbb{R}^D, b \in \mathbb{R} \), and \( \sigma: \mathbb{R} \rightarrow \mathbb{R} \), can naturally be extended to a multi-dimension output by applying a similar transformation to every output

\[
\mathbb{R}^D \rightarrow \mathbb{R}^C \\
\quad x \mapsto \sigma(wx + b),
\]

with \( w \in \mathbb{R}^{C \times D}, b \in \mathbb{R}^C \), and \( \sigma \) is applied component-wise.
Even though it has no practical value implementation-wise, we can represent such a model as a combination of units. More importantly, we can extend it.

\[
\sigma_x f(x; w, b) w, b
\]

Single unit

One layer of units

Multiple layers of units
This latter structure can be formally defined, with $x^{(0)} = x$,

$$\forall l = 1, \ldots, L, \ x^{(l)} = \sigma \left( w^{(l)} x^{(l-1)} + b^{(l)} \right)$$

and $f(x; w, b) = x^{(L)}$.

Such a model is a Multi-Layer Perceptron (MLP).
Note that if $\sigma$ is an affine transformation, the full MLP is a composition of affine mappings, and itself an affine mapping.

Consequently:

⚠️ **The activation function $\sigma$ should be non-linear**, or the resulting MLP is an affine mapping with a peculiar parametrization.
The two classical activation functions are the hyperbolic tangent

\[ x \mapsto \frac{2}{1 + e^{-2x}} - 1 \]

and the rectified linear unit (ReLU)

\[ x \mapsto \max(0, x) \]
Universal approximation
We can approximate any $\psi \in \mathcal{C}([a, b], \mathbb{R})$ with a linear combination of translated/scaled ReLU functions.

$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \ldots$$

This is true for other activation functions under mild assumptions.
Extending this result to any $\psi \in C([0,1]^D, \mathbb{R})$ requires a bit of work.

We can approximate the sin function with the previous scheme, and use the density of Fourier series to get the final result:

$$\forall \epsilon > 0, \exists K, w \in \mathbb{R}^K \times D, b \in \mathbb{R}^K, \omega \in \mathbb{R}^K \text{ s.t.} \quad \max_{x \in [0,1]^D} |\psi(x) - \omega \cdot \sigma(w \cdot x + b) | \leq \epsilon$$
So we can approximate any continuous function

$$\psi : [0, 1]^D \rightarrow \mathbb{R}$$

with a one hidden layer perceptron

$$x \mapsto \omega \cdot \sigma(w \cdot x + b),$$

where $b \in \mathbb{R}^K$, $w \in \mathbb{R}^{K \times D}$, and $\omega \in \mathbb{R}^K$.

This is the universal approximation theorem.
A better approximation requires a larger hidden layer (larger $K$), and this theorem says nothing about the relation between the two.

So this results states that we can make the **training error** as low as we want by using a larger hidden layer. It states nothing about the **test error**

Deploying MLP in practice is often a balancing act between under-fitting and over-fitting.