EE-559 – Deep learning

6.1. Benefits of depth

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For image classification for instance, there has been a trend toward deeper architectures to improve performance.

Network	Nb. layers
LeNet5 (leCun et al., 1998)	5
AlexNet (Krizhevsky et al., 2012)	8
VGG (Simonyan and Zisserman, 2014)	11–19
GoogleLeNet (Szegedy et al., 2015)	22
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A theoretical analysis provides an intuition of how a network's output "irregularity" grows linearly with its width and exponentially with its depth.

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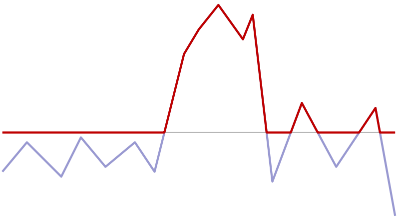

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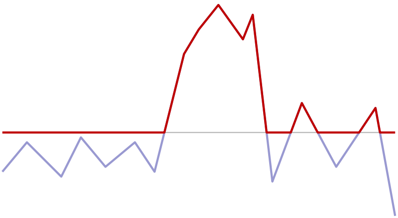

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$$\forall f \in \mathscr{F},\ \kappa(\sigma(f)) \leq 2\kappa(f)\,,$$

and we also have

$$\forall (f, g) \in \mathscr{F}^2,\ \kappa(f + g) \leq \kappa(f) + \kappa(g)\,.$$

Consider a MLP with ReLU, a single input unit, and a single output unit.

$$x_1^0 = x,$$

$$\forall d = 1, \dots, D, \forall i, \quad \begin{cases} s_i^d & = \sum_{j=1}^{W^{d-1}} w_{i,j}^d x_j^{d-1} + b_i^d \\ x_i^d & = \sigma(s_i^d) \end{cases}$$

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$$\forall l, i, \kappa\left(x_i^l\right) = \kappa\left(\sigma(s_i^l)\right) \leq 2\kappa\left(s_i^l\right) \leq 2 \sum_{j=1}^{W_{l-1}} \kappa\left(x_j^{l-1}\right)$$

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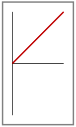
and we get the following bound for any ReLU MLP

$$\kappa(y) \leq 2^D \prod_{d=1}^{D} W_d.$$

Although this seems quite a pessimist bound, we can hand-design a network that [almost] reaches it:



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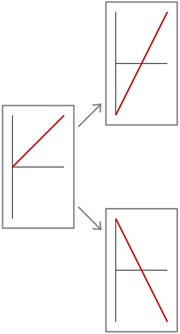

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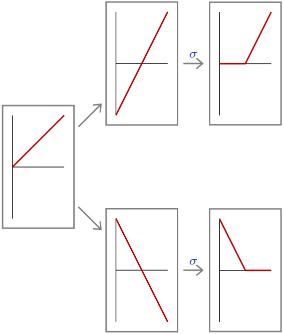

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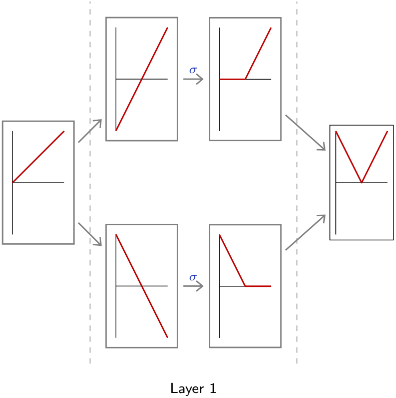

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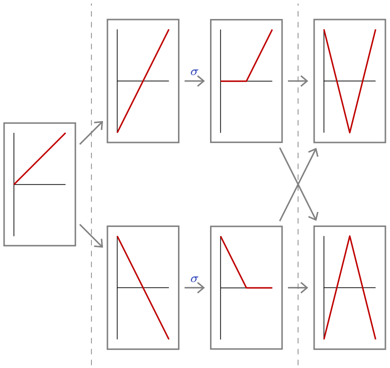

Layer 1

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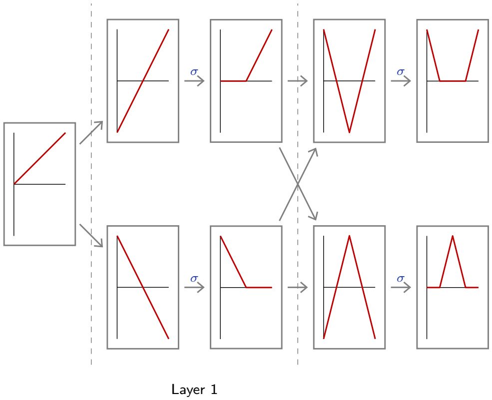

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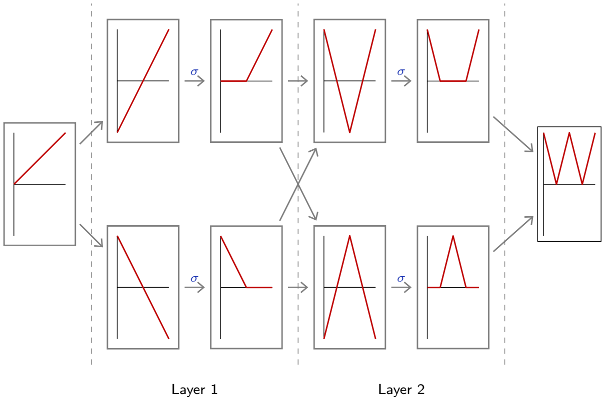

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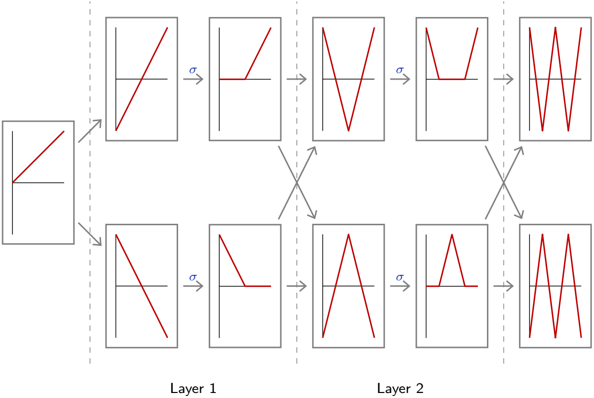

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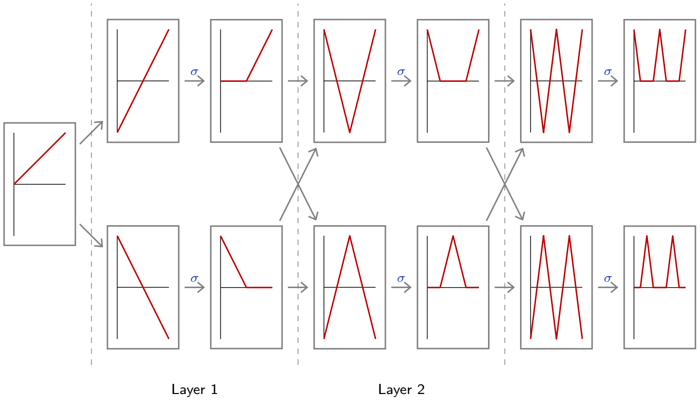

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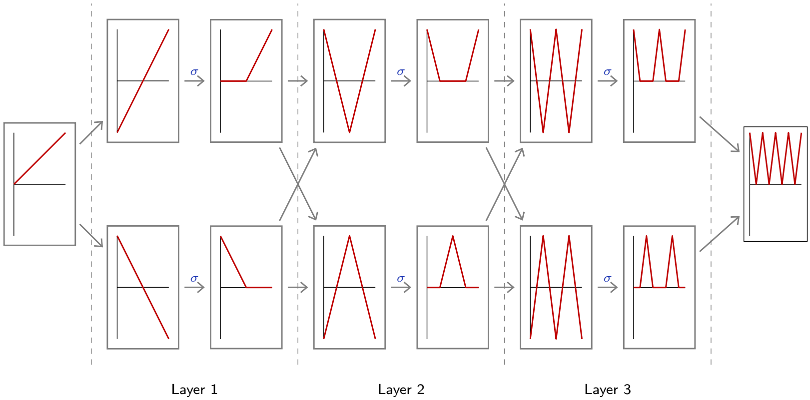
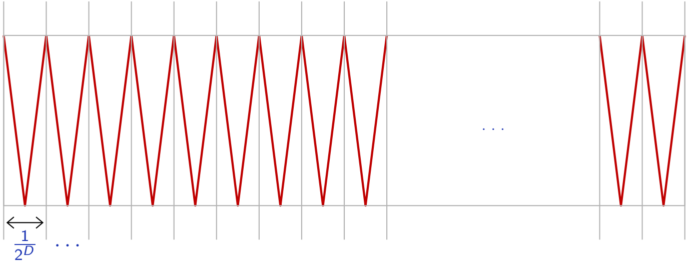
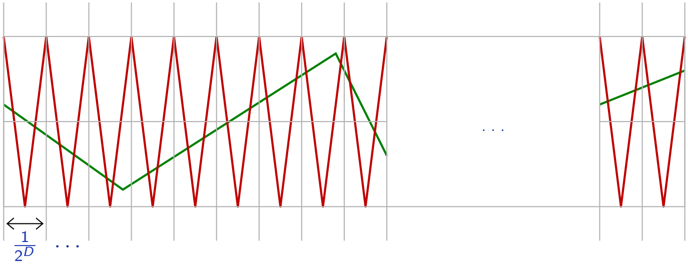


So for any D, there is a network with D hidden layers and $2D$ hidden units which computes an $f : [0, 1] \rightarrow [0, 1]$ of period $1/2^D$





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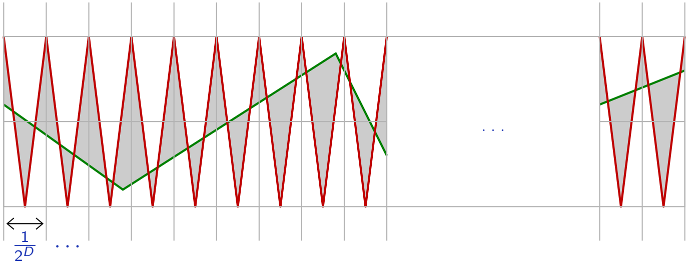

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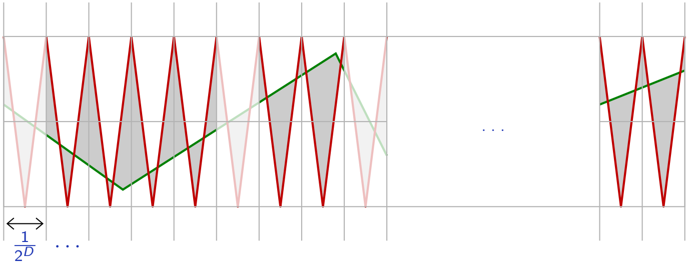
$$\int_0^1 |f(x) - g(x)|$$



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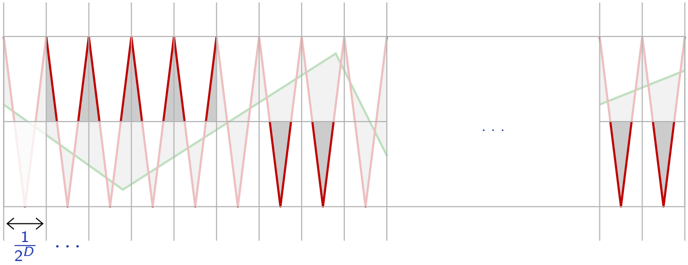

Given $g \in \mathscr{F}$, it crosses $\frac{1}{2}$ at most $\kappa(g)$ times, which means that on at least $2^D - \kappa(g)$ segments of length $1/2^D$, it is on one side of $\frac{1}{2}$, and

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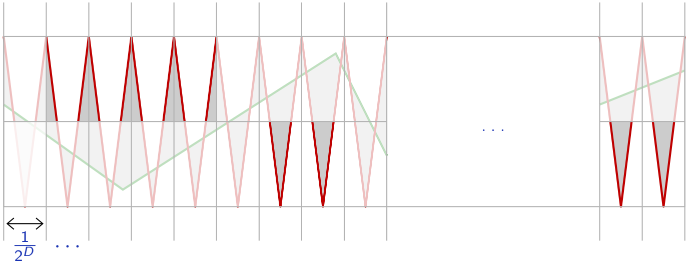

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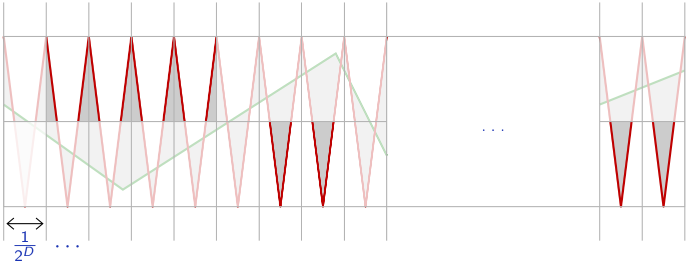

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And we multiply f by 16 to get our final result.

So, considering ReLU MLPs with a single input/output:

There exists a network f with D^* layers, and $2D^*$ internal units, such that, for any network g with D layers of sizes $\{W_1, \dots, W_D\}$:

$$\|f - g\|_1 \geq 1 - \frac{2^D}{2^{D^*}} \prod_{d=1}^{D} W_d.$$

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This is a simplified variant of results by Telgarsky (2015, 2016).

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In particular we have to ensure that

- the gradient does not “vanish” (Bengio et al., 1994; Hochreiter et al., 2001),
- gradient amplitude is homogeneous so that all parts of the network train at the same rate (Glorot and Bengio, 2010),
- the gradient does not vary too unpredictably when the weights change (Balduzzi et al., 2017).

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An additional issue for training very large architectures is the computational cost, which often turns out to be the main practical problem.

The end

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