

EE-559 – Deep learning

4.4. Convolutions

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<https://fleuret.org/ee559/>

Mon Oct 15 08:59:41 UTC 2018

If they were handled as normal “unstructured” vectors, large-dimension signals such as sound samples or images would require models of intractable size.

For instance a linear layer taking a 256×256 RGB image as input, and producing an image of same size would require

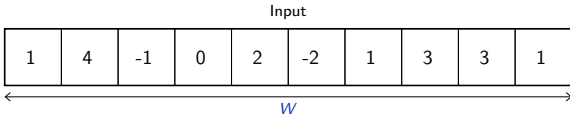
$$(256 \times 256 \times 3)^2 \simeq 3.87e+10$$

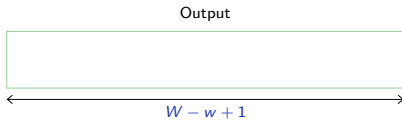
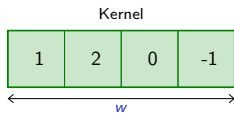
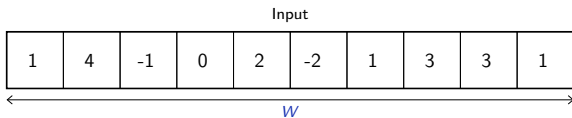
parameters, with the corresponding memory footprint ($\simeq 150\text{Gb}$!), and excess of capacity.

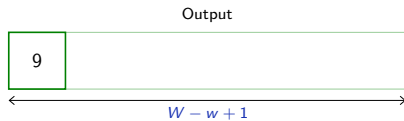
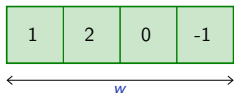
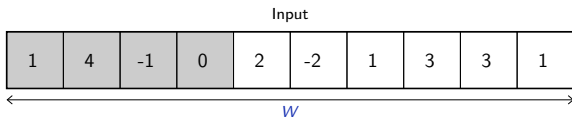
Moreover, this requirement is inconsistent with the intuition that such large signals have some “invariance in translation”. **A representation meaningful at a certain location can / should be used everywhere.**

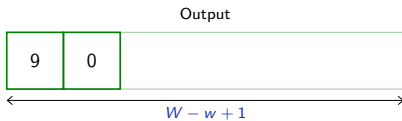
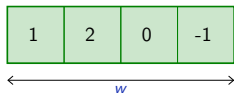
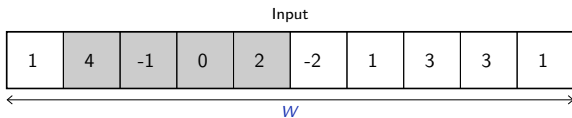
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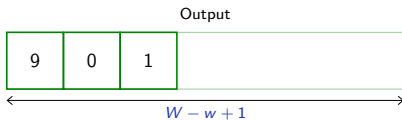
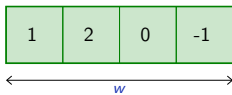
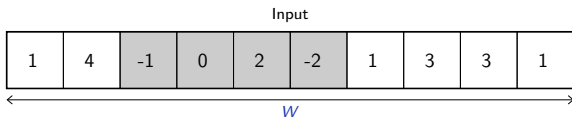
A convolution layer embodies this idea. It applies the same linear transformation locally, everywhere, and preserves the signal structure.

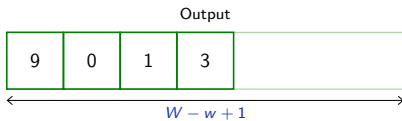
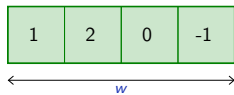
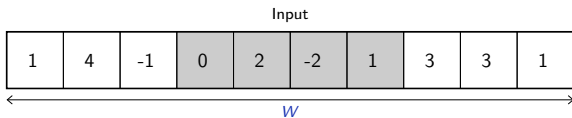


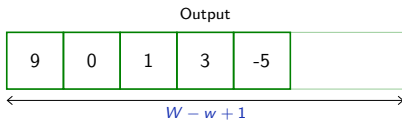
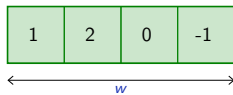
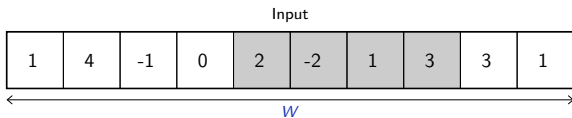


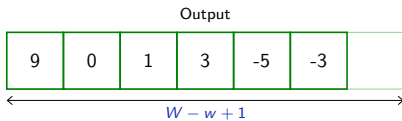
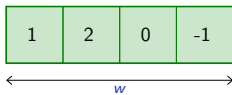
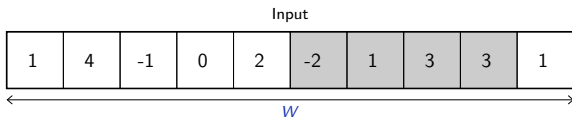


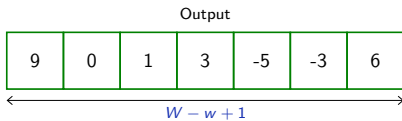
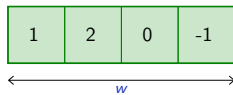
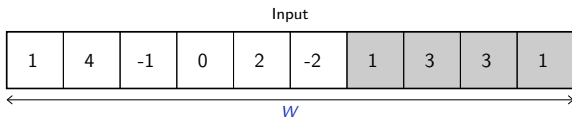


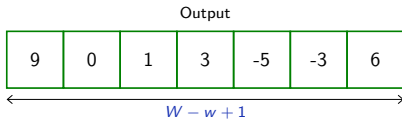
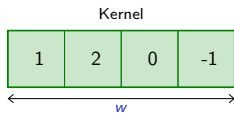
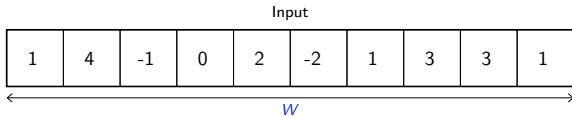












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the convolution $x \circledast u$ is a vector of size $W - w + 1$, with

$$\begin{aligned}(x \circledast u)_i &= \sum_{j=1}^w x_{i-1+j} u_j \\ &= (x_i, \dots, x_{i+w-1}) \cdot u\end{aligned}$$

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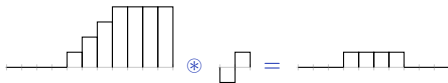
This differs from the usual convolution since the kernel and the signal are both visited in increasing index order.

Convolution can implement in particular differential operators, e.g.

$$(0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4) \circledast (-1, 1) = (0, 0, 0, 1, 1, 1, 1, 0, 0, 0).$$

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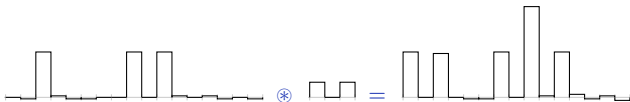


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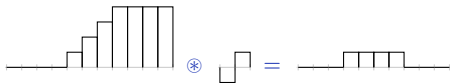


or crude “template matcher”, e.g.

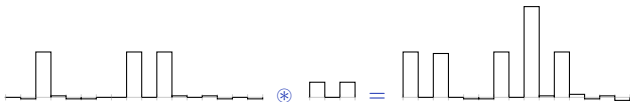


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Both of these computation examples are indeed “invariant by translation”.

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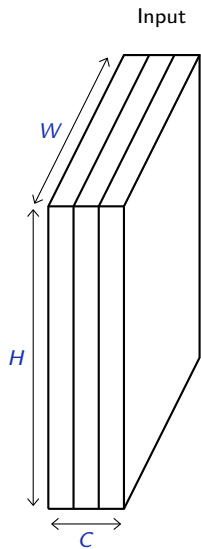
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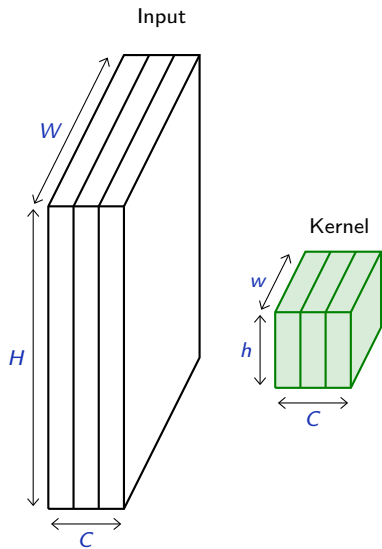
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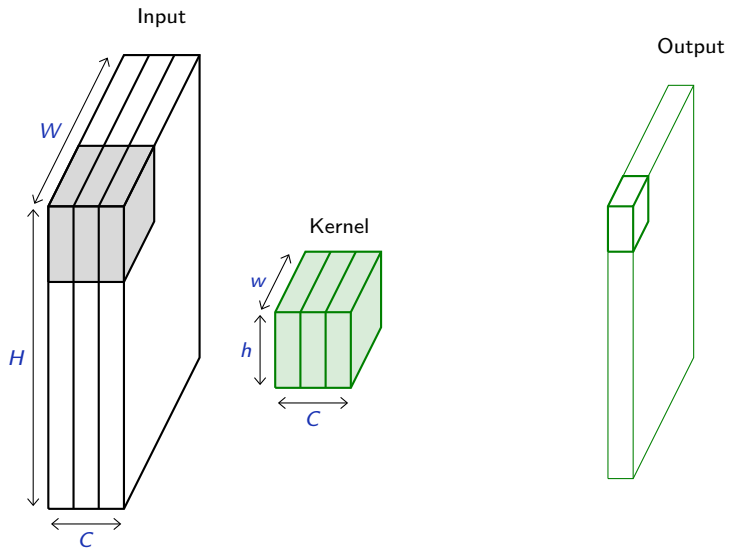


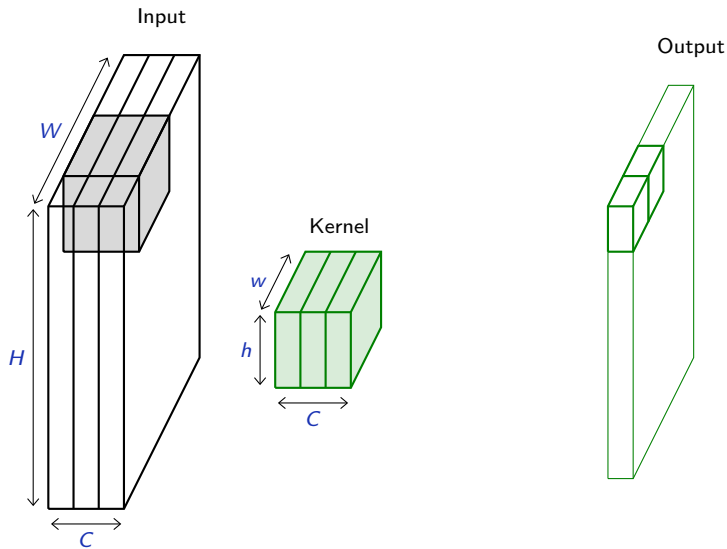
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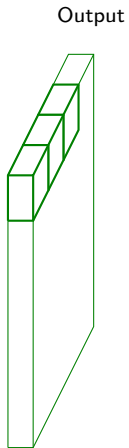
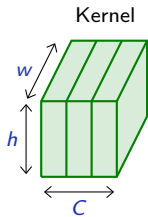
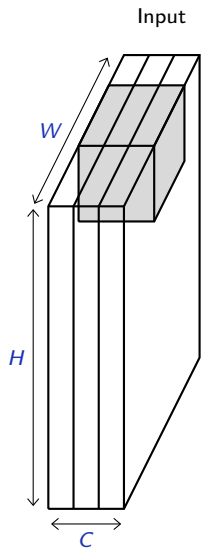
In a standard convolution layer, D such convolutions are combined to generate a $D \times (H - h + 1) \times (W - w + 1)$ output.

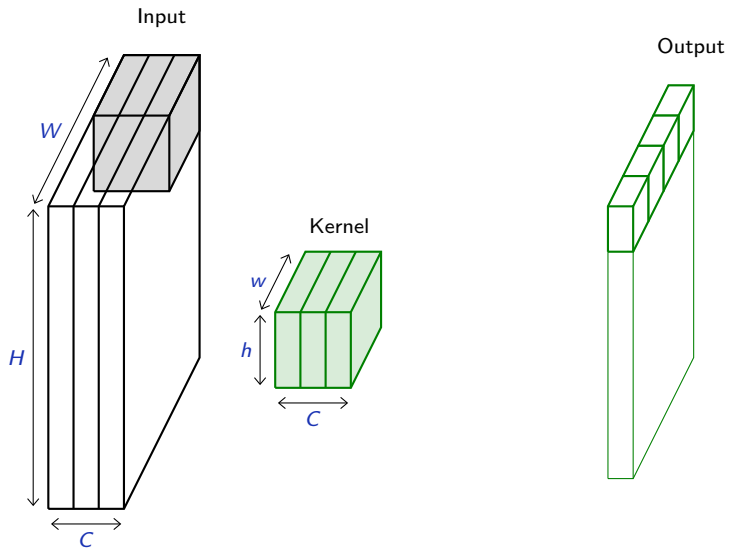


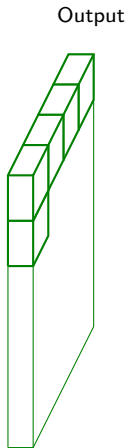
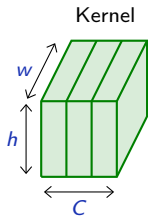
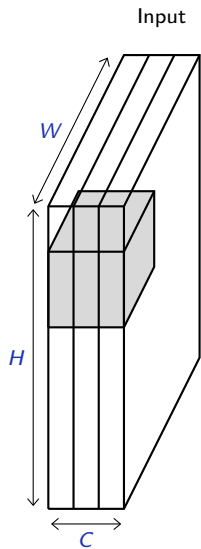


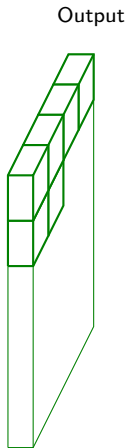
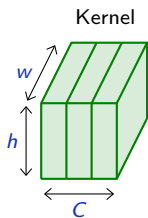
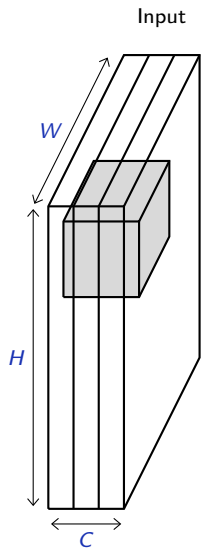


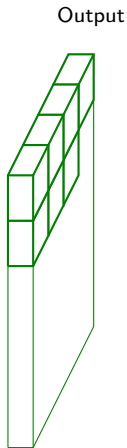
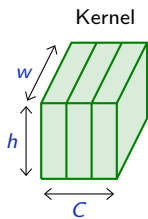
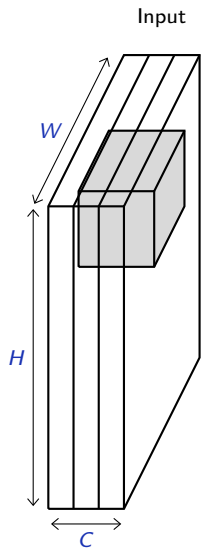


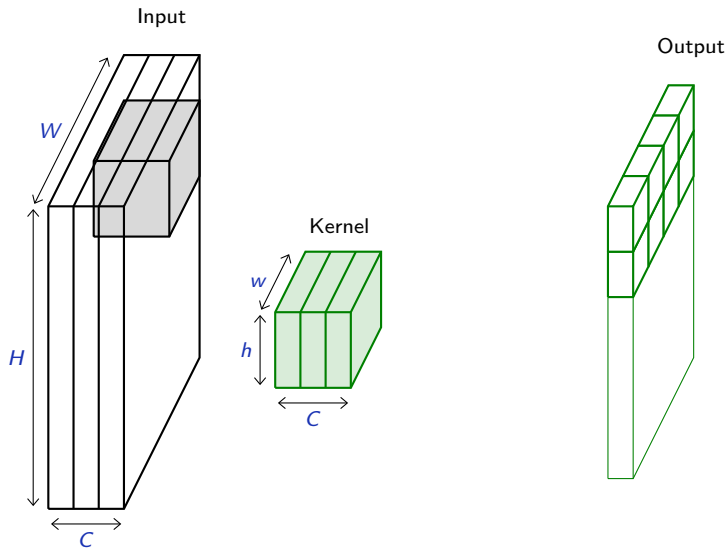


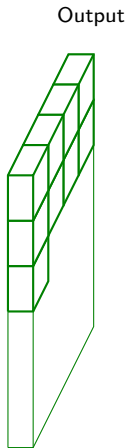
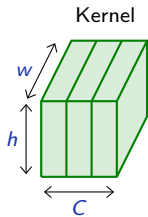
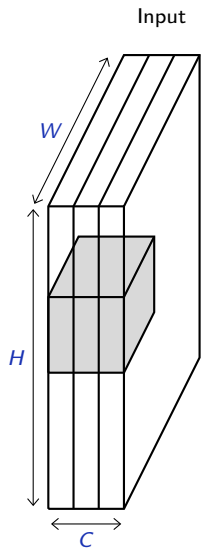


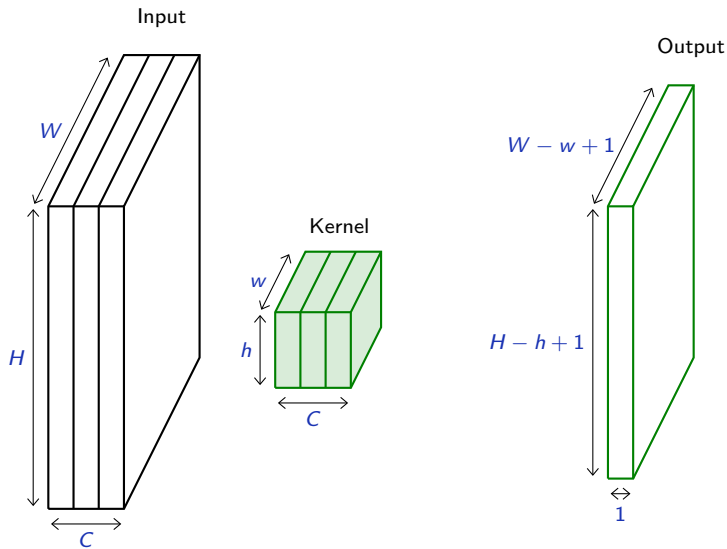


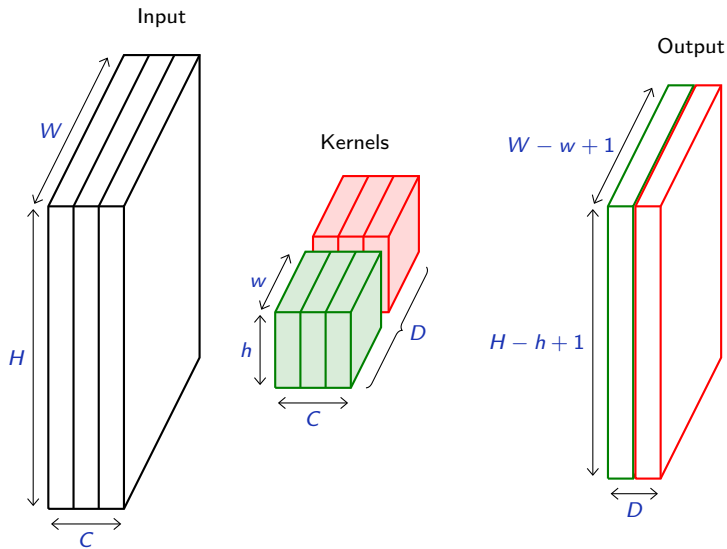












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A 1d signal is converted into a 1d signal, a 2d signal into a 2d, and neighboring parts of the input signal influence neighboring parts of the output signal.

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A 3d convolution can be used if the channel index has some metric meaning, such as time for a series of grayscale video frames. Otherwise swiping across channels makes no sense.

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In the context of convolutional networks, a standard linear layer is called a **fully connected layer** since every input influences every output.

```
torch.nn.functional.conv2d(input, weight, bias=None,  
                             stride=1, padding=0, dilation=1, groups=1)
```

Implements a 2d convolution, where `weight` contains the kernels, and is $D \times C \times h \times w$, `bias` is of dimension D , `input` is of dimension

$$N \times C \times H \times W$$

and the result is of dimension

$$N \times D \times (H - h + 1) \times (W - w + 1).$$

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```
>>> weight = torch.empty(5, 4, 2, 3).normal_()
>>> bias = torch.empty(5).normal_()
>>> input = torch.empty(117, 4, 10, 3).normal_()
>>> output = torch.nn.functional.conv2d(input, weight, bias)
>>> output.size()
torch.Size([117, 5, 9, 1])
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Similar functions implement 1d and 3d convolutions.

```
x = mnist_train.train_data[12].float().view(1, 1, 28, 28)

weight = torch.empty(5, 1, 3, 3)

weight[0, 0] = torch.tensor([ [ 0., 0., 0. ],
                              [ 0., 1., 0. ],
                              [ 0., 0., 0. ] ])

weight[1, 0] = torch.tensor([ [ 1., 1., 1. ],
                              [ 1., 1., 1. ],
                              [ 1., 1., 1. ] ])

weight[2, 0] = torch.tensor([ [ -1., 0., 1. ],
                              [ -1., 0., 1. ],
                              [ -1., 0., 1. ] ])

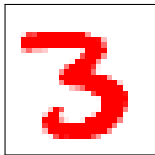
weight[3, 0] = torch.tensor([ [ -1., -1., -1. ],
                              [ 0., 0., 0. ],
                              [ 1., 1., 1. ] ])

weight[4, 0] = torch.tensor([ [ 0., -1., 0. ],
                              [ -1., 4., -1. ],
                              [ 0., -1., 0. ] ])

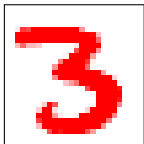
y = torch.nn.functional.conv2d(x, weight)
```

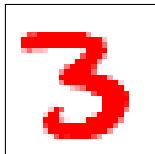
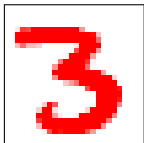
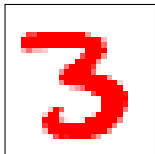


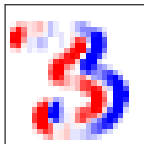
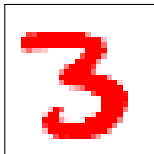
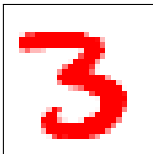
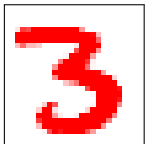
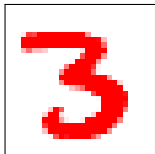
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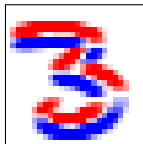
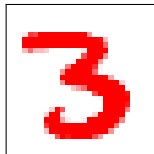
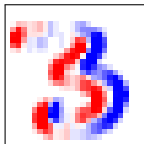
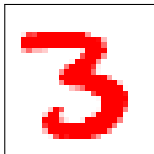
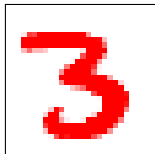
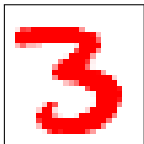
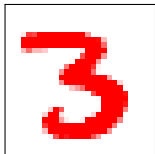


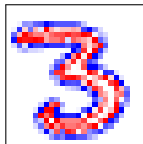
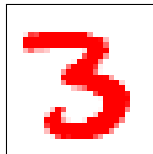
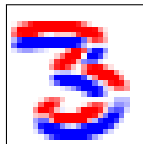
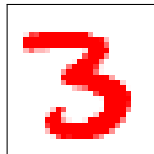
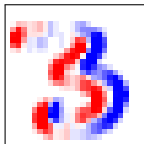
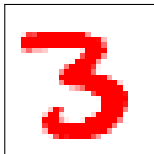
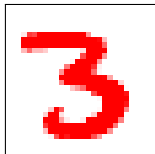
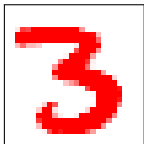
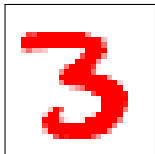
=











```
class torch.nn.Conv2d(in_channels, out_channels,  
                      kernel_size, stride=1, padding=0, dilation=1,  
                      groups=1, bias=True)
```

Wraps the convolution into a `Module`, with the kernels and biases as `Parameter` properly randomized at creation.

The kernel size is either a pair (h, w) or a single value k interpreted as (k, k) .

```
class torch.nn.Conv2d(in_channels, out_channels,
                      kernel_size, stride=1, padding=0, dilation=1,
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```

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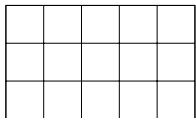
```
>>> f = nn.Conv2d(in_channels = 4, out_channels = 5, kernel_size = (2, 3))
>>> for n, p in f.named_parameters(): print(n, p.size())
...
weight torch.Size([5, 4, 2, 3])
bias torch.Size([5])
>>> x = torch.empty(117, 4, 10, 3).normal_()
>>> y = f(x)
>>> y.size()
torch.Size([117, 5, 9, 1])
```

Padding and stride

Convolutions have two additional standard parameters:

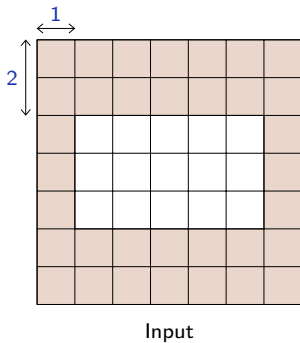
- The **padding** specifies the size of a zeroed frame added around the input,
- the **stride** specifies a step size when moving the kernel across the signal.

Here with $C \times 3 \times 5$ as input

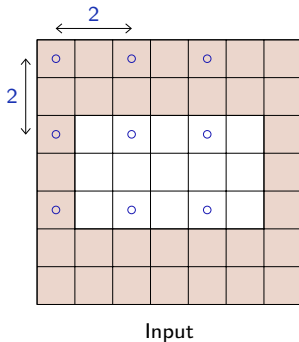


Input

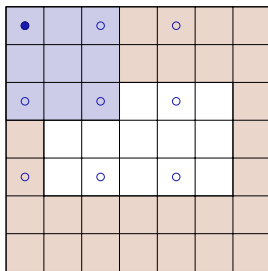
Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$



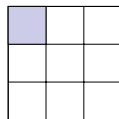
Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$



Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

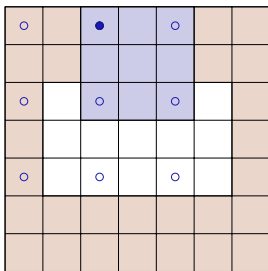


Input

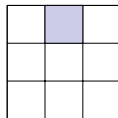


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

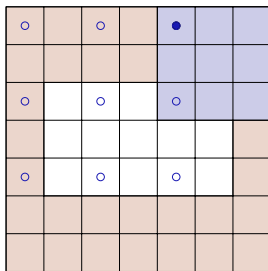


Input

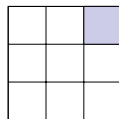


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

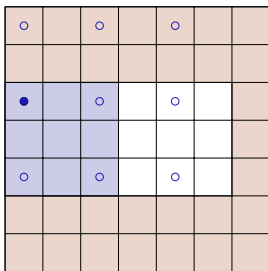


Input

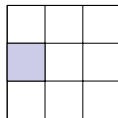


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

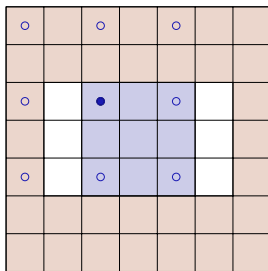


Input

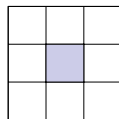


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

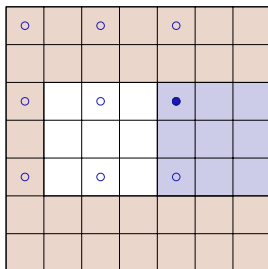


Input

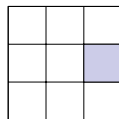


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

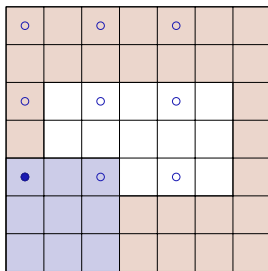


Input

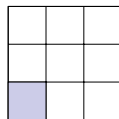


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

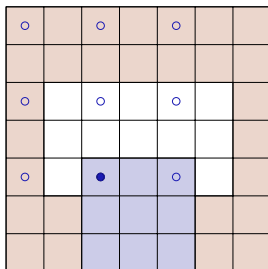


Input

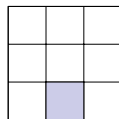


Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$

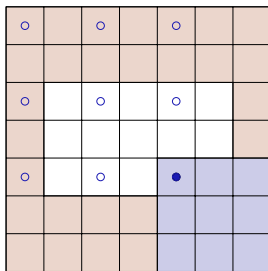


Input

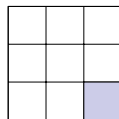


Output

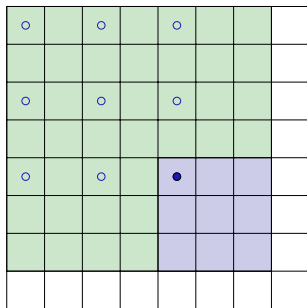
Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$, the output is $1 \times 3 \times 3$.



Input



Output



A convolution with a stride greater than 1 may not cover the input map completely, hence may ignore some of the input values.

Dilated convolution

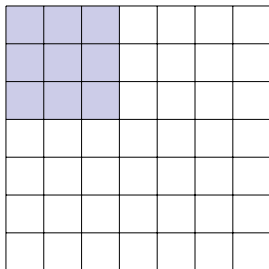
Convolution operations admit one more standard parameter that we have not discussed yet: The dilation, which modulates the expansion of the filter support (Yu and Koltun, 2015).

It is 1 for standard convolutions, but can be greater, in which case the resulting operation can be envisioned as a convolution with a regularly sparsified filter.

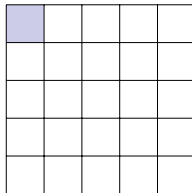
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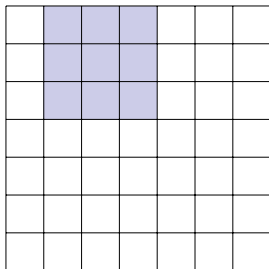
This notion comes from signal processing, where it is referred to as *algorithme à trous*, hence the term sometime used of “convolution à trous”.



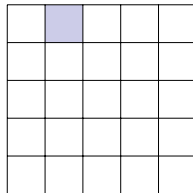
Input



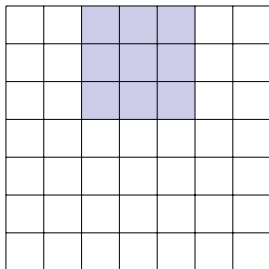
Output



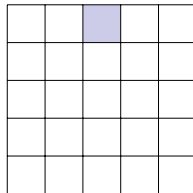
Input



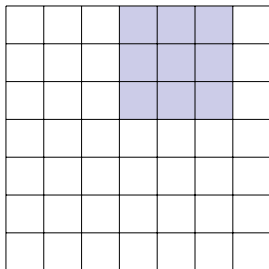
Output



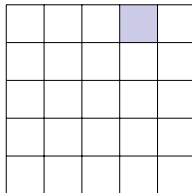
Input



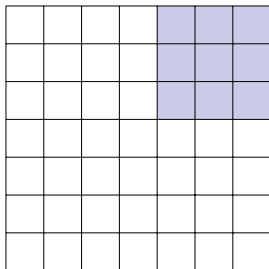
Output



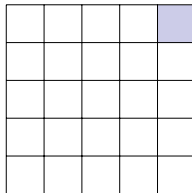
Input



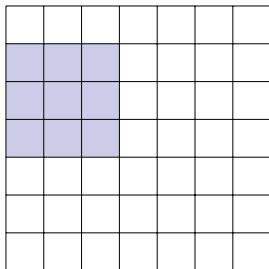
Output



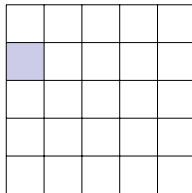
Input



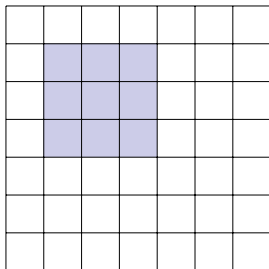
Output



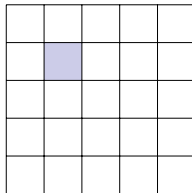
Input



Output

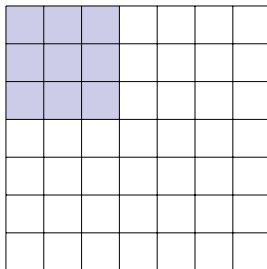


Input

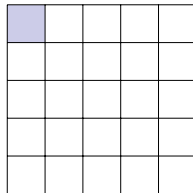


Output

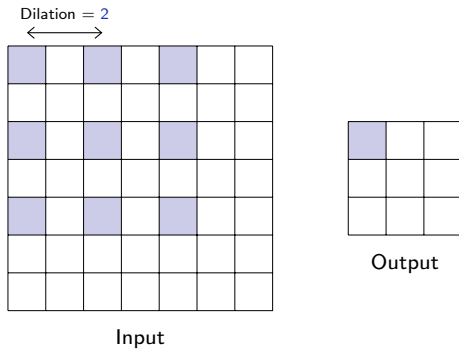
Dilation = 1

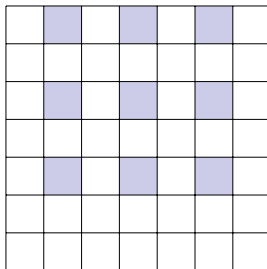


Input

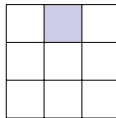


Output

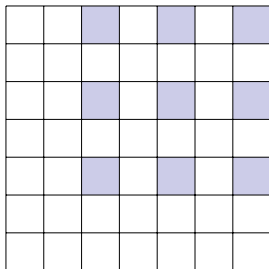




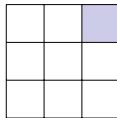
Input



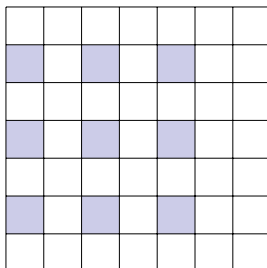
Output



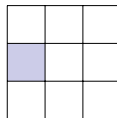
Input



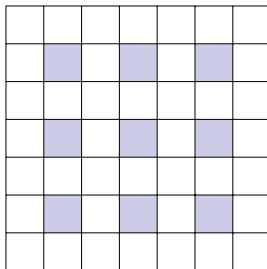
Output



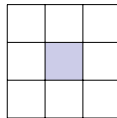
Input



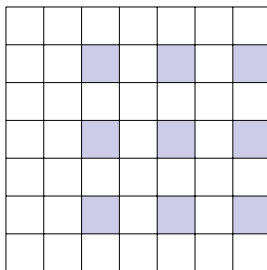
Output



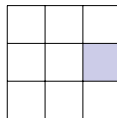
Input



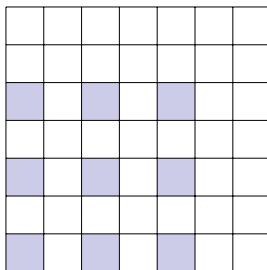
Output



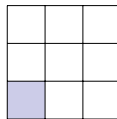
Input



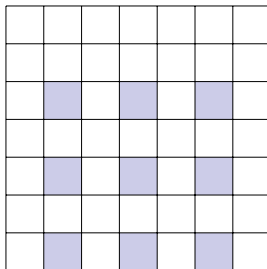
Output



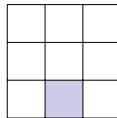
Input



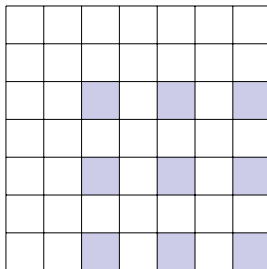
Output



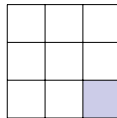
Input



Output



Input



Output

A convolution with a 1d kernel of size k and dilation d can be interpreted as a convolution with a filter of size $1 + (k - 1)d$ with only k non-zero coefficients.

For with $k = 3$ and $d = 4$, the difference between the input map size and the output map size is $1 + (3 - 1)4 - 1 = 8$.

```
>>> x = torch.empty(1, 1, 20, 30).normal_()
>>> l = nn.Conv2d(1, 1, kernel_size = 3, dilation = 4)
>>> l(x).size()
torch.Size([1, 1, 12, 22])
```

Having a dilation greater than one increases the units' receptive field size without increasing the number of parameters.

Convolutions with stride or dilation strictly greater than one reduce the activation map size, for instance to make a final classification decision.

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Such networks have the advantage of simplicity:

- non-linear operations are only in the activation function,
- joint operations that combine multiple activations to produce one are only in linear layers.

The end

References

- F. Yu and V. Koltun. Multi-scale context aggregation by dilated convolutions. *CoRR*, abs/1511.07122v3, 2015.