EE-559 – Deep learning

3.2. Probabilistic view of a linear classifier

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Consider the following class populations

$$\forall y \in \{0,1\}, x \in \mathbb{R}^D,$$
$$\mu_{X \mid Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} \exp\left(-\frac{1}{2}(x - m_y)\Sigma^{-1}(x - m_y)^T\right).$$

That is, they are Gaussian with **the same covariance matrix Σ.** This is the **homoscedasticity** assumption.

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&= \frac{\mu_{X \mid Y=1}(x) P(Y = 1)}{\mu_{X \mid Y=0}(x) P(Y = 0) + \mu_{X \mid Y=1}(x) P(Y = 1)}
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&= \frac{1}{1+\frac{\mu_{X|Y=0}(x)}{\mu_{X|Y=1}(x)}\frac{P(Y=0)}{P(Y=1)}}.
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It follows that, with

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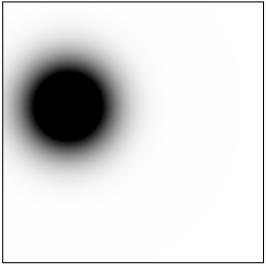
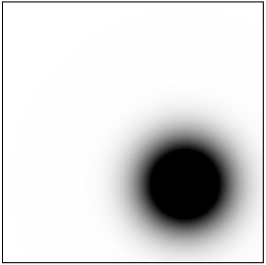
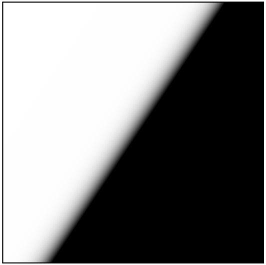
The homoscedasticity makes the second-order terms vanish.



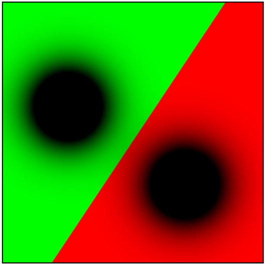
$\mu_{X|Y=0}$

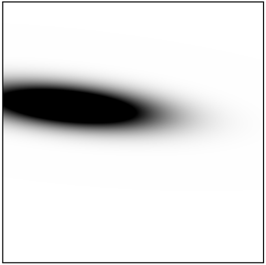


$\mu_{X|Y=1}$



$P(Y = 1 \mid X = x)$





$\mu_{X \mid Y=0}$

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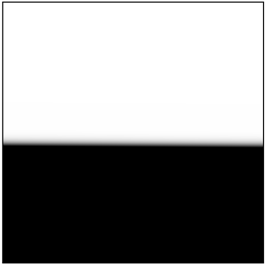

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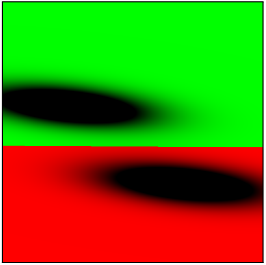

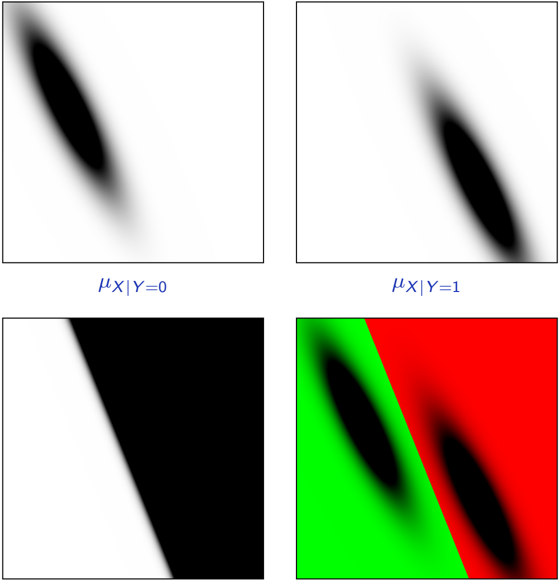

$\mu_{X|Y=0}$ $\mu_{X|Y=1}$

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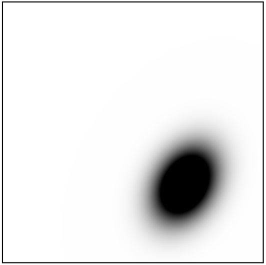
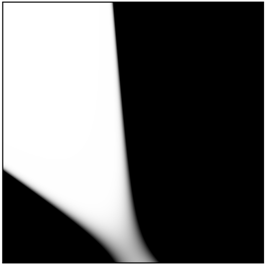


$\mu_{X|Y=1}$



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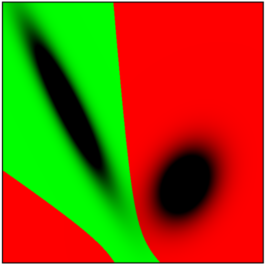
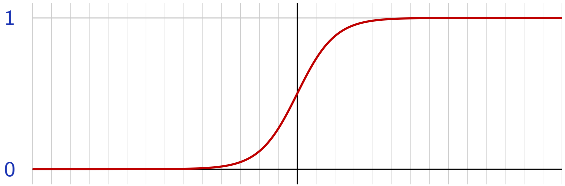


Note that the (logistic) sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}},$$

looks like a "soft heavyside"



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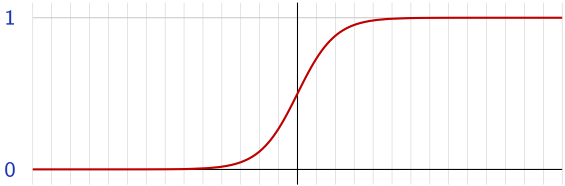

So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.

We can use the model from LDA

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First, to simplify the next slide, note that we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x),$$

hence if Y takes value in $\{-1, 1\}$ then

$$\forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)).$$

We have

$$
\begin{aligned}
\log \mu_{W,B}(w, b \mid \mathscr{D} = \mathbf{d}) \\
&= \log \frac{\mu_{\mathscr{D}}(\mathbf{d} \mid W = w, B = b)\, \mu_{W,B}(w, b)}{\mu_{\mathscr{D}}(\mathbf{d})} \\
&= \log \mu_{\mathscr{D}}(\mathbf{d} \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z \\
&= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'
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$$= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'$$

This is the **logistic regression**, whose loss aims at minimizing

$$-\log \sigma(y_n f(x_n)).$$

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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.

The end