Deep learning

7.1. Transposed convolutions

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https://fleuret.org/ee559/

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Constructing deep generative architectures requires layers to increase the signal dimension, the contrary of what we have done so far with feed-forward networks.
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Some generative processes optimize the input, and as such rely on back-propagation to expend the signal from a low-dimension representation to the high-dimension signal space.
Constructing deep generative architectures requires layers to increase the signal dimension, the contrary of what we have done so far with feed-forward networks.

Some generative processes optimize the input, and as such rely on back-propagation to expend the signal from a low-dimension representation to the high-dimension signal space.

The same can be done in the forward pass with transposed convolution layers whose forward operation corresponds to a convolution layer’s backward pass.
Consider a 1d convolution with a kernel $\kappa$:

\[
y_i = (x \ast \kappa)_i \\
= \sum_{a} x_{i+a-1} \kappa_{a} \\
= \sum_{u} x_{u} \kappa_{u-i+1}.
\]
Consider a 1d convolution with a kernel \( \kappa \)

\[
y_i = (x \odot \kappa)_i
\]

\[
= \sum_a x_{i+a-1} \kappa_a
\]

\[
= \sum_u x_u \kappa_{u-i+1}.
\]

We get

\[
\left[ \frac{\partial \ell}{\partial x} \right]_u = \frac{\partial \ell}{\partial x_u}
\]

\[
= \sum_i \frac{\partial \ell}{\partial y_i} \frac{\partial y_i}{\partial x_u}
\]

\[
= \sum_i \frac{\partial \ell}{\partial y_i} \kappa_{u-i+1}.
\]

which looks a lot like a standard convolution layer, except that the kernel coefficients are visited in reverse order.
This is actually the standard convolution operator from signal processing. If $*$ denotes this operation, we have

$$(x * \kappa)_i = \sum_a x_a \kappa_{i-a+1}.$$
This is actually the standard convolution operator from signal processing. If $*$ denotes this operation, we have

$$(x * \kappa)_i = \sum_a x_a \kappa_{i-a+1}.$$ 

Coming back to the backward pass of the convolution layer, if

$$y = x \bigodot \kappa$$

then

$$\begin{bmatrix} \frac{\partial \ell}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ell}{\partial y} \end{bmatrix} * \kappa.$$
In the deep-learning field, since it corresponds to transposing the weight matrix of the equivalent fully-connected layer, it is called a **transposed convolution**.

\[
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 & 0 & 0 \\
0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 & 0 \\
0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 \\
0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 \\
0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 \\
0 & 0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3
\end{pmatrix}^\top
= 
\begin{pmatrix}
\kappa_1 & 0 & 0 & 0 & 0 \\
\kappa_2 & \kappa_1 & 0 & 0 & 0 \\
\kappa_3 & \kappa_2 & \kappa_1 & 0 & 0 \\
0 & \kappa_3 & \kappa_2 & \kappa_1 & 0 \\
0 & 0 & \kappa_3 & \kappa_2 & \kappa_1 \\
0 & 0 & 0 & \kappa_3 & \kappa_2 \\
0 & 0 & 0 & 0 & \kappa_3
\end{pmatrix}
\]
In the deep-learning field, since it corresponds to transposing the weight matrix of the equivalent fully-connected layer, it is called a **transposed convolution**.

\[
\begin{pmatrix}
\kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 & 0 & 0 \\
0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 & 0 \\
0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 & 0 \\
0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 & 0 \\
0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 \\
0 & 0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 \\
\end{pmatrix}
\top
= 
\begin{pmatrix}
\kappa_1 & 0 & 0 & 0 & 0 & 0 \\
\kappa_2 & \kappa_1 & 0 & 0 & 0 \\
\kappa_3 & \kappa_2 & \kappa_1 & 0 & 0 \\
0 & \kappa_3 & \kappa_2 & \kappa_1 & 0 \\
0 & 0 & \kappa_3 & \kappa_2 & \kappa_1 \\
0 & 0 & 0 & \kappa_3 & \kappa_2 \\
0 & 0 & 0 & 0 & \kappa_3 \\
\end{pmatrix}
\]

A convolution can be seen as a series of inner products, a transposed convolution can be seen as a weighted sum of translated kernels.
Convolution layer

Input

\[
\begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1 \\
\end{array}
\]

\[W\]
Convolution layer

Input

| 1 | 4 | -1 | 0 | 2 | -2 | 1 | 3 | 3 | 1 |

Kernel

| 1 | 2 | 0 | -1 |

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Convolution layer

Input

\[
\begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1 \\
\end{array}
\]

\[W\]

\[w\]

Output

\[
\begin{array}{c}
9 \\
\end{array}
\]

\[W - w + 1\]
Convolution layer

Input

$$\begin{bmatrix}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1
\end{bmatrix}$$

Output

$$\begin{bmatrix}
9 & 0
\end{bmatrix}$$

$$W - w + 1$$
Convolution layer

Input

\[
\begin{bmatrix}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1
\end{bmatrix}
\]

\[W\]

Kernel

\[
\begin{bmatrix}
1 & 2 & 0 & -1
\end{bmatrix}
\]

\[w\]

Output

\[
\begin{bmatrix}
9 & 0 & 1
\end{bmatrix}
\]

\[W - w + 1\]
Convolution layer

Input

\[ \begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1 \\
\end{array} \]

Kernel

\[ \begin{array}{cccc}
1 & 2 & 0 & -1 \\
\end{array} \]

Output

\[ \begin{array}{cccc}
9 & 0 & 1 & 3 \\
\end{array} \]

\[ W - w + 1 \]
Convolution layer

Input

\[
\begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1
\end{array}
\]

\(W - w + 1\)

Output

\[
\begin{array}{cccc}
9 & 0 & 1 & 3 & -5
\end{array}
\]
Convolution layer

Input

| 1 | 4 | -1 | 0 | 2 | -2 | 1 | 3 | 3 | 1 |

Output

| 9 | 0 | 1 | 3 | -5 | -3 |

$W - w + 1$
Convolution layer

Input

\[
\begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1 \\
\end{array}
\]

\(W\) \(w\) \(W - w + 1\)

Output

\[
\begin{array}{cccccccc}
9 & 0 & 1 & 3 & -5 & -3 & 6 \\
\end{array}
\]
Convolution layer

Input

\[
\begin{array}{cccccccc}
1 & 4 & -1 & 0 & 2 & -2 & 1 & 3 & 3 & 1 \\
\end{array}
\]

Kernel

\[
\begin{array}{cccc}
1 & 2 & 0 & -1 \\
\end{array}
\]

Output

\[
\begin{array}{cccccccc}
9 & 0 & 1 & 3 & -5 & -3 & 6 \\
\end{array}
\]
Transposed convolution layer

Input

\[
\begin{bmatrix}
2 & 3 & 0 & -1
\end{bmatrix}
\]

Kernel

\[
\begin{bmatrix}
1 & 2 & -1
\end{bmatrix}
\]
Transposed convolution layer

Input

\[
\begin{bmatrix}
2 & 3 & 0 & -1
\end{bmatrix}
\]

Kernel

\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 4 & -2
\end{bmatrix}
\]

Output

\[
\begin{bmatrix}
2
\end{bmatrix}
\]

\[W + w - 1\]
Transposed convolution layer

Input

\[ \begin{array}{cccc}
2 & 3 & 0 & -1 \\
\end{array} \]

Kernel

\[ \begin{array}{ccc}
1 & 2 & -1 \\
2 & 4 & -2 \\
3 & 6 & -3 \\
\end{array} \]

Output

\[ \begin{array}{cc}
2 & 7 \\
\end{array} \]

\[ W + w - 1 \]
Transposed convolution layer

Input

\[
\begin{bmatrix}
2 & 3 & 0 & -1
\end{bmatrix}
\]

\[W\]

Kernel

\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 4 & -2 \\
3 & 6 & -3 \\
0 & 0 & 0
\end{bmatrix}
\]

Output

\[
\begin{bmatrix}
2 & 7 & 4
\end{bmatrix}
\]

\[W + w - 1\]
Transposed convolution layer

Input

\[ \begin{bmatrix}
   2 & 3 & 0 & -1
\end{bmatrix} \]

Kernel

\[ \begin{bmatrix}
   1 & 2 & -1
\end{bmatrix} \]

+ \[ \begin{bmatrix}
   2 & 4 & -2 \\
   3 & 6 & -3 \\
   0 & 0 & 0 \\
   -1 & -2 & 1
\end{bmatrix} \]

Output

\[ \begin{bmatrix}
   2 & 7 & 4 & -4 & -2 & 1
\end{bmatrix} \]
Transposed convolution layer

Input

\[
\begin{pmatrix}
2 & 3 & 0 & -1
\end{pmatrix}
\]

\[W\]

Output

\[
\begin{pmatrix}
2 & 7 & 4 & -4 & -2 & 1
\end{pmatrix}
\]

\[W + w - 1\]
Transposed convolution layer

Input

\[
\begin{bmatrix}
2 & 3 & 0 & -1 \\
\end{bmatrix}
\]

Kernel

\[
\begin{bmatrix}
1 & 2 & -1 \\
\end{bmatrix}
\]

Output

\[
\begin{bmatrix}
2 & 7 & 4 & -4 & -2 & 1 \\
\end{bmatrix}
\]
F.conv_transpose1d implements the operation we just described. It takes as input a batch of multi-channel samples, and produces a batch of multi-channel samples.

```python
>>> x = torch.tensor([[0., 0., 1., 0., 0., 0., 0.]])
>>> k = torch.tensor([[1., 2., 3.]])
>>> F.conv1d(x, k)
tensor([[ 3., 2., 1., 0., 0.]])
```

```
>>> F.conv_transpose1d(x, k)
tensor([[ 0., 0., 1., 2., 3., 0., 0., 0., 0.]])
```

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\texttt{F.conv_transpose1d} implements the operation we just described. It takes as input a batch of multi-channel samples, and produces a batch of multi-channel samples.

\begin{verbatim}
>>> x = torch.tensor([[0., 0., 1., 0., 0., 0., 0.]])
>>> k = torch.tensor([[1., 2., 3.]])
>>> F.conv1d(x, k)
tensor([[ 3., 2., 1., 0., 0.]])

\begin{align*}
\ast & = \\
\end{align*}

\begin{verbatim}
>>> F.conv_transpose1d(x, k)
tensor([[ 0., 0., 1., 2., 3., 0., 0., 0., 0.]])
\end{verbatim}
\end{verbatim}
The class `nn.ConvTranspose1d` embeds that operation into a `nn.Module`.

```python
g>>> x = torch.tensor([[ 1., 0., 0., 0., -1.]])
>>> m = nn.ConvTranspose1d(1, 1, kernel_size=3)
>>> with torch.autograd.no_grad():
...     m.bias.zero_()
...     m.weight.copy_(torch.tensor([ 1, 2, 1 ]))
...
Parameter containing:
tensor([0.], requires_grad=True)
Parameter containing:
tensor([[1., 2., 1.]], requires_grad=True)
```
Transposed convolutions also have a \textit{dilation} parameter that behaves as for convolution and expends the kernel size without increasing the number of parameters by making it sparse.
Transposed convolutions also have a \textit{dilation} parameter that behaves as for convolution and expends the kernel size without increasing the number of parameters by making it sparse.

They also have a \textit{stride} and \textit{padding} parameters, however, due to the relation between convolutions and transposed convolutions:

While for convolutions \textit{stride} and \textit{padding} are defined in the input map, for transposed convolutions these parameters are defined in the output map, and the latter modulates a cropping operation.
Transposed convolution layer (stride = 2)

\[ \text{Output} = s(W - 1) + w \]
Transposed convolution layer (stride = 2)

Input

\[ \begin{array}{cccc}
2 & 3 & 0 & -1 \\
\end{array} \]

Kernel

\[ \begin{array}{ccc}
1 & 2 & -1 \\
2 & 4 & -2 \\
\end{array} \]

Output

\[ \begin{array}{cc}
2 & 4 \\
\end{array} \]

\[ s(W - 1) + w \]
Transposed convolution layer (stride = 2)

\[ s(W - 1) + w \]
Transposed convolution layer (stride = 2)

Input

\[
\begin{array}{c c c c}
2 & 3 & 0 & -1 \\
\end{array}
\]

Kernel

\[
\begin{array}{c c c c}
1 & 2 & -1 \\
\end{array}
\]

\[
\begin{array}{c c c c}
2 & 4 & -2 \\
\end{array}
\]

\[
\begin{array}{c c c c}
3 & 6 & -3 \\
\end{array}
\]

\[
\begin{array}{c c c c}
0 & 0 & 0 \\
\end{array}
\]

Output

\[
\begin{array}{c c c c c c}
2 & 4 & 1 & 6 & -3 & 0 \\
\end{array}
\]

\[
\text{s}(W - 1) + w
\]
Transposed convolution layer (stride = 2)

Input

\[
\begin{array}{ccc}
2 & 3 & 0 \\
\end{array}
\]

Output

\[
\begin{array}{cccccccc}
2 & 4 & 1 & 6 & -3 & 0 & -1 & -2 & 1 \\
\end{array}
\]

Kernel

\[
\begin{array}{ccc}
1 & 2 & -1 \\
\end{array}
\]
Transposed convolution layer (stride = 2)

Input

\[
\begin{array}{cccc}
2 & 3 & 0 & -1 \\
\end{array}
\]

\[\leftarrow W \rightarrow\]

Kernel

\[
\begin{array}{cccc}
2 & 4 & -2 \\
3 & 6 & -3 \\
0 & 0 & 0 \\
-1 & -2 & 1 \\
\end{array}
\]

\[\leftarrow s \rightarrow\]

Output

\[
\begin{array}{cccccccc}
2 & 4 & 1 & 6 & -3 & 0 & -1 & -2 & 1 \\
\end{array}
\]

\[\leftarrow s(W - 1) + w \rightarrow\]
Transposed convolution layer (stride = 2)

Input

\[
\begin{array}{cccc}
2 & 3 & 0 & -1 \\
\end{array}
\]

Kernel

\[
\begin{array}{ccc}
1 & 2 & -1 \\
\end{array}
\]

Output

\[
\begin{array}{cccccccc}
2 & 4 & 1 & 6 & -3 & 0 & -1 & -2 & 1 \\
\end{array}
\]

\[s(W - 1) + w\]
The composition of a convolution and a transposed convolution of same parameters keep the signal size [roughly] unchanged.

⚠️ A convolution with a stride greater than one may ignore parts of the signal. Its composition with the corresponding transposed convolution generates a map of the size of the observed area.
The composition of a convolution and a transposed convolution of same parameters keep the signal size [roughly] unchanged.

A convolution with a stride greater than one may ignore parts of the signal. Its composition with the corresponding transposed convolution generates a map of the size of the observed area.

For instance, a 1d convolution of kernel size $w$ and stride $s$ composed with the transposed convolution of same parameters maintains the signal size $W$, only if

$$\exists q \in \mathbb{N}, \; W = w + s \cdot q.$$
It has been observed that transposed convolutions may create some grid-structure artifacts, since generated pixels are not all covered similarly.

For instance with a $4 \times 4$ kernel and stride 3
An alternative is to use an analytic up-scaling, implemented in the PyTorch functional `F.interpolate`.

```python
>>> x = torch.tensor([[[ 1., 2. ], [ 3., 4. ]]])
>>> F.interpolate(x, scale_factor = 3, mode = 'bilinear')
tensor([[[1.0000, 1.0000, 1.3333, 1.6667, 2.0000, 2.0000],
[1.0000, 1.0000, 1.3333, 1.6667, 2.0000, 2.0000],
[1.6667, 1.6667, 2.0000, 2.3333, 2.6667, 2.6667],
[2.3333, 2.3333, 2.6667, 3.0000, 3.3333, 3.3333],
[3.0000, 3.0000, 3.3333, 3.6667, 4.0000, 4.0000],
[3.0000, 3.0000, 3.3333, 3.6667, 4.0000, 4.0000]]])
```

```python
>>> F.interpolate(x, scale_factor = 3, mode = 'nearest')
tensor([[[1., 1., 1., 2., 2., 2.],
[1., 1., 1., 2., 2., 2.],
[1., 1., 1., 2., 2., 2.],
[3., 3., 3., 4., 4., 4.],
[3., 3., 3., 4., 4., 4.],
[3., 3., 3., 4., 4., 4.]]])
```
An alternative is to use an analytic up-scaling, implemented in the PyTorch functional `F.interpolate`.

```python
>>> x = torch.tensor([[[[ 1., 2. ], [ 3., 4. ]]]])
>>> F.interpolate(x, scale_factor = 3, mode = 'bilinear')
tensor([[[[1.0000, 1.0000, 1.3333, 1.6667, 2.0000, 2.0000],
         [1.0000, 1.0000, 1.3333, 1.6667, 2.0000, 2.0000],
         [1.6667, 1.6667, 2.0000, 2.3333, 2.6667, 2.6667],
         [2.3333, 2.3333, 2.6667, 3.0000, 3.3333, 3.3333],
         [3.0000, 3.0000, 3.3333, 3.6667, 4.0000, 4.0000],
         [3.0000, 3.0000, 3.3333, 3.6667, 4.0000, 4.0000]]]])
```

```python
>>> F.interpolate(x, scale_factor = 3, mode = 'nearest')
tensor([[[[1., 1., 1., 2., 2., 2.],
         [1., 1., 1., 2., 2., 2.],
         [1., 1., 1., 2., 2., 2.],
         [3., 3., 3., 4., 4., 4.],
         [3., 3., 3., 4., 4., 4.],
         [3., 3., 3., 4., 4., 4.]]]])
```
Such module is usually combined with a convolution to learn local corrections to undesirable artifacts of the up-scaling.

In practice, a transposed convolution such as

```python
import torch.nn as nn

nic, noc = 3, 1
kernel_size = 3, stride = 2,
padding = 1, output_padding = 1),

tconv = nn.ConvTranspose2d(nic, noc,
                          kernel_size = 3, stride = 2,
                          padding = 1, output_padding = 1),

y = tconv(x)
```
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In practice, a transposed convolution such as

```python
tconv = nn.ConvTranspose2d(nic, noc,
    kernel_size = 3, stride = 2,
    padding = 1, output_padding = 1),

y = tconv(x)
```

can be replaced by

```python
conv = nn.Conv2d(nic, noc, kernel_size = 3, padding = 1)

u = F.interpolate(x, scale_factor = 2, mode = 'bilinear')

y = conv(u)
```
The end