Deep learning

6.1. Benefits of depth

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Using deeper architectures has been key in improving performance in many applications. For instance image classification:

model	top-1 err.	top-5 err.
VGG-16 [41]	28.07	9.33
GoogLeNet [44]	-	9.15
PReLU-net [13]	24.27	7.38
plain-34	28.54	10.02
ResNet-34 A	25.03	7.76
ResNet-34 B	24.52	7.46
ResNet-34 C	24.19	7.40
ResNet-50	22.85	6.71
ResNet-101	21.75	6.05
ResNet-152	**21.43**	**5.71**

Table 3. Error rates (%, **10-crop** testing) on ImageNet validation. VGG-16 is based on our test. ResNet-50/101/152 are of option B that only uses projections for increasing dimensions.

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"Notably, we did not depart from the classical ConvNet architecture of LeCun et al. (1989), but improved it by substantially increasing the depth."

(Simonyan and Zisserman, 2014)

A theoretical analysis provides an intuition of how a network's output "irregularity" grows:

- linearly with its width and
- exponentially with its depth.

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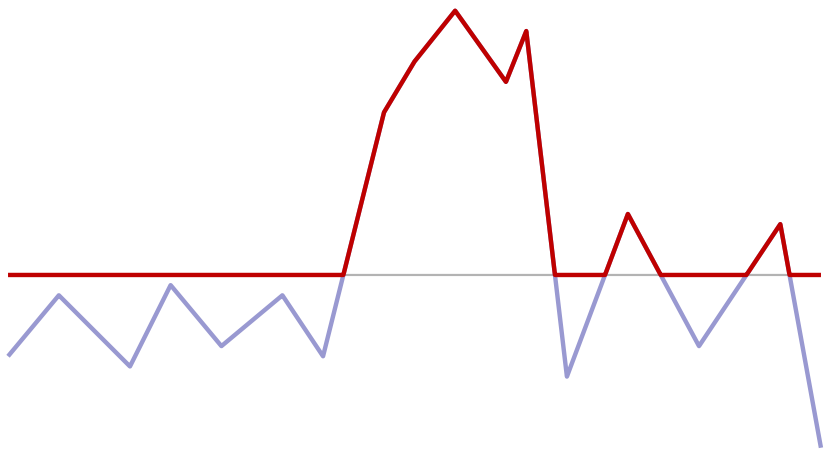

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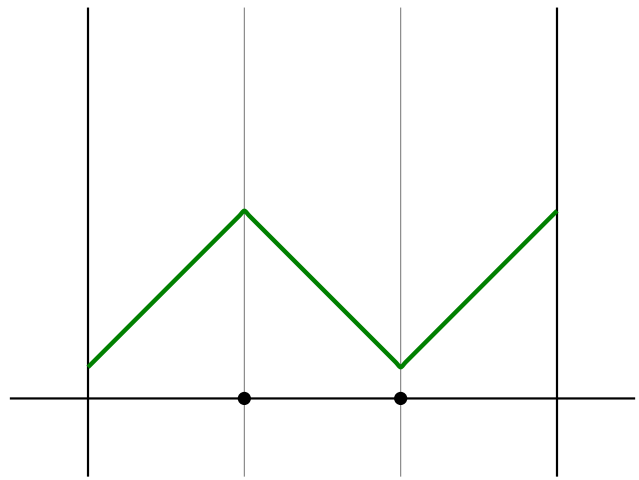
If we compose σ and $f \in \mathscr{F}$, any linear piece that does not cross 0 remains a single piece or disappears, and one that does cross 0 breaks into two, hence

$$\forall f \in \mathscr{F},\ \kappa(\sigma(f)) \leq 2\kappa(f)\,.$$

Also, when summing functions, a change of slop in the sum happens only if there was a change of slop in one of the operands.



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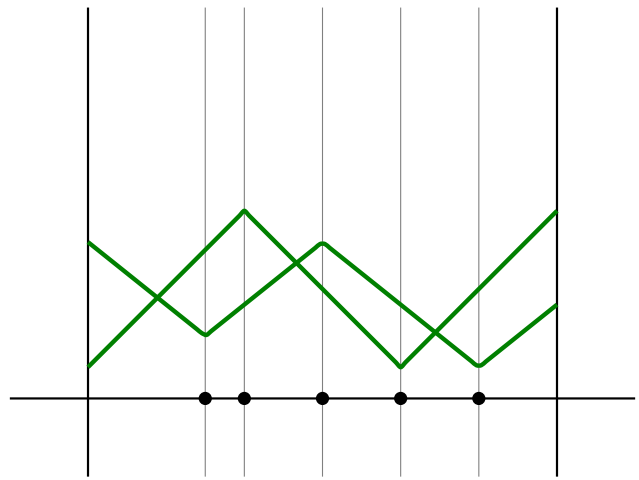

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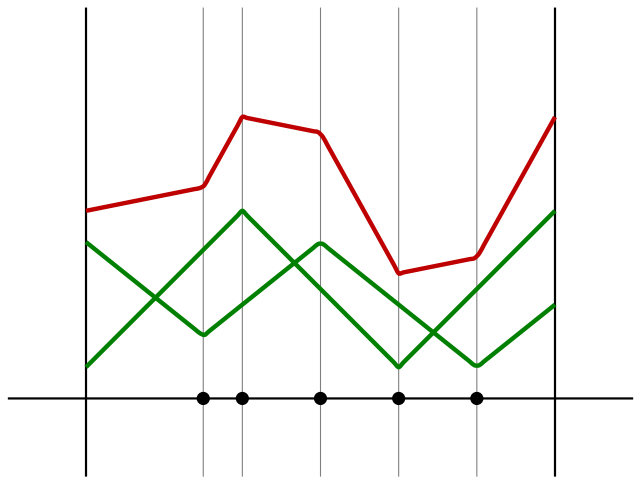

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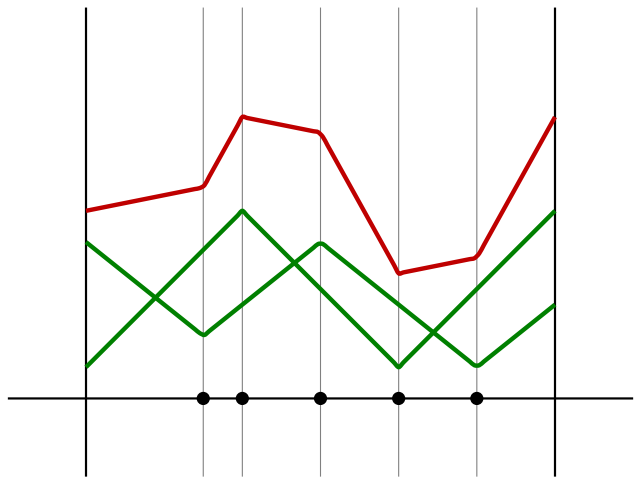

Hence

$$\forall f_n \in \mathscr{F}, n = 1, \ldots, N, \ \kappa\left(\sum_n f_n\right) \leq \sum_n \kappa(f_n)\,.$$

Consider a MLP with ReLU, a single input unit, and a single output unit.

$$x_1^0 = x,$$

$$\forall d = 1, \ldots, D, \forall i, \quad \left\{ \begin{array}{ll} s_i^d & = \sum_{j=1}^{W^{(d-1)}} w_{i,j}^d x_j^{d-1} + b_i^d \\ x_i^d & = \sigma(s_i^d) \end{array} \right.$$

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All the s_i^ds and x_i^ds are piece-wise linear functions of x with $\forall i, \kappa\left(s_i^1\right) = 1$, and

$$\forall d, i, \kappa\left(x_i^d\right) = \kappa\left(\sigma(s_i^d)\right) \leq 2\kappa\left(s_i^d\right) \leq 2 \sum_{j=1}^{W^{(d-1)}} \kappa\left(x_j^{d-1}\right)$$

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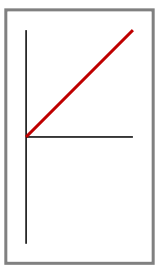
$$\forall d, \max_i \kappa\left(x_i^d\right) \leq 2W^{(d-1)} \max_j \kappa\left(x_j^{d-1}\right)$$

and we get the following bound for any ReLU MLP

$$\kappa(y) \leq 2^D \prod_{d=1}^{D} W^{(d)}.$$

Although this seems quite a pessimist bound, we can hand-design a network that [almost] reaches it:



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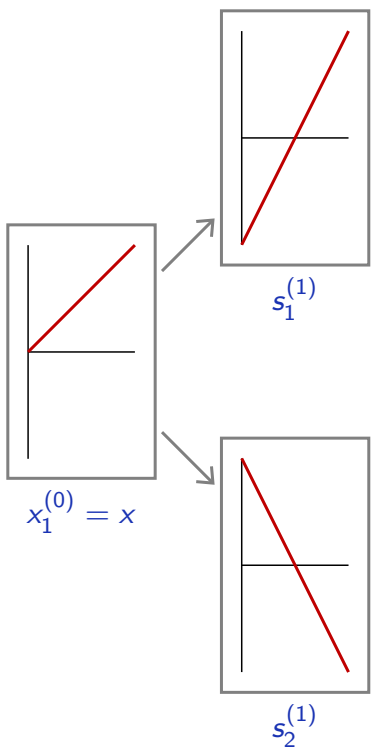

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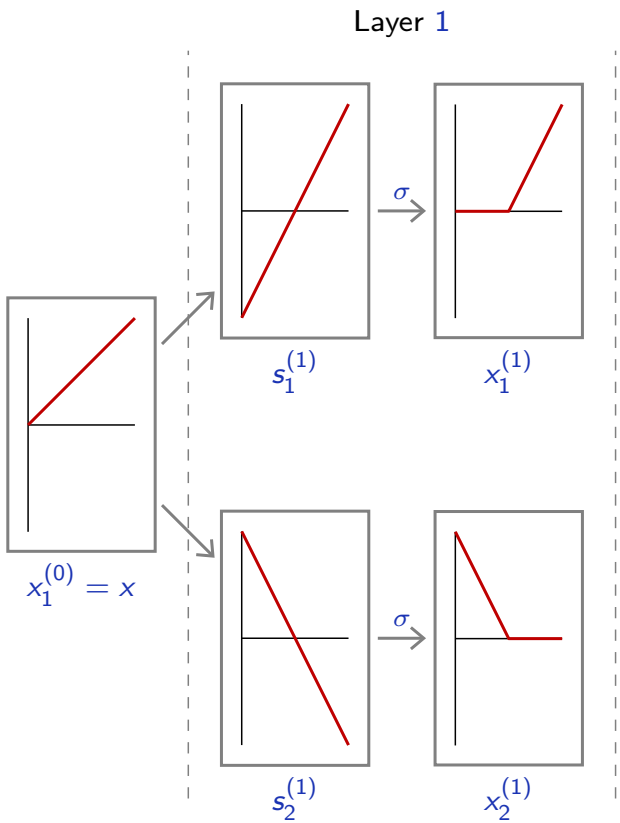

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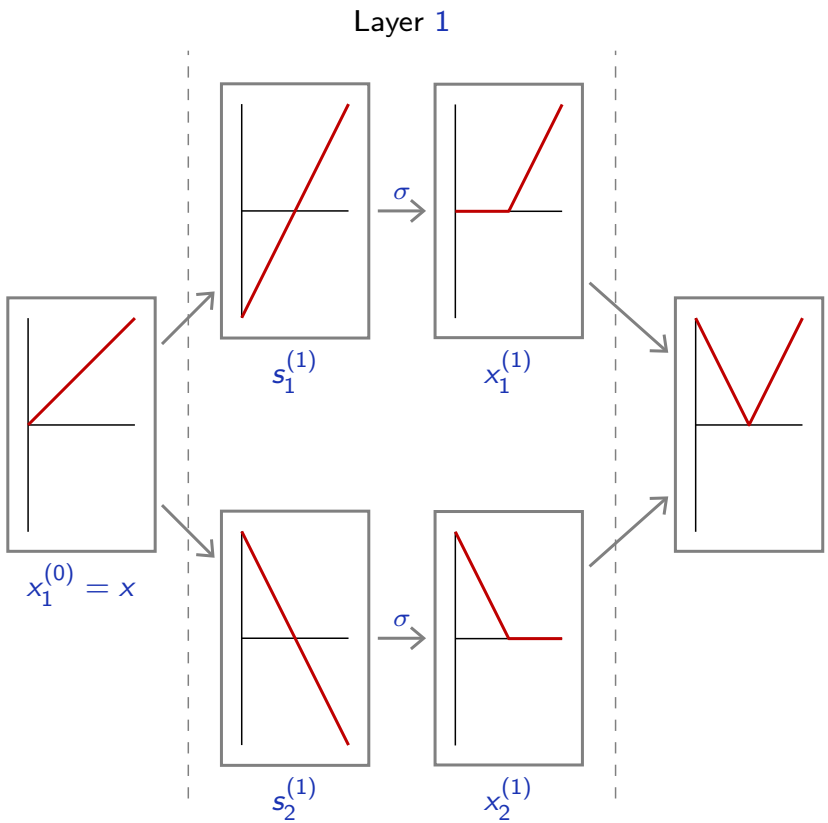

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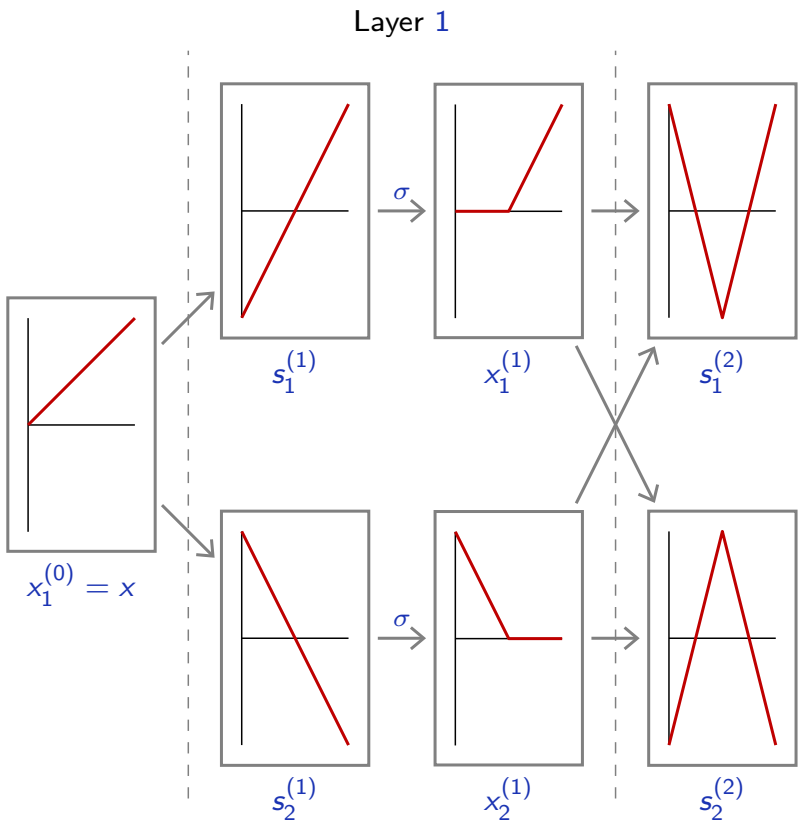

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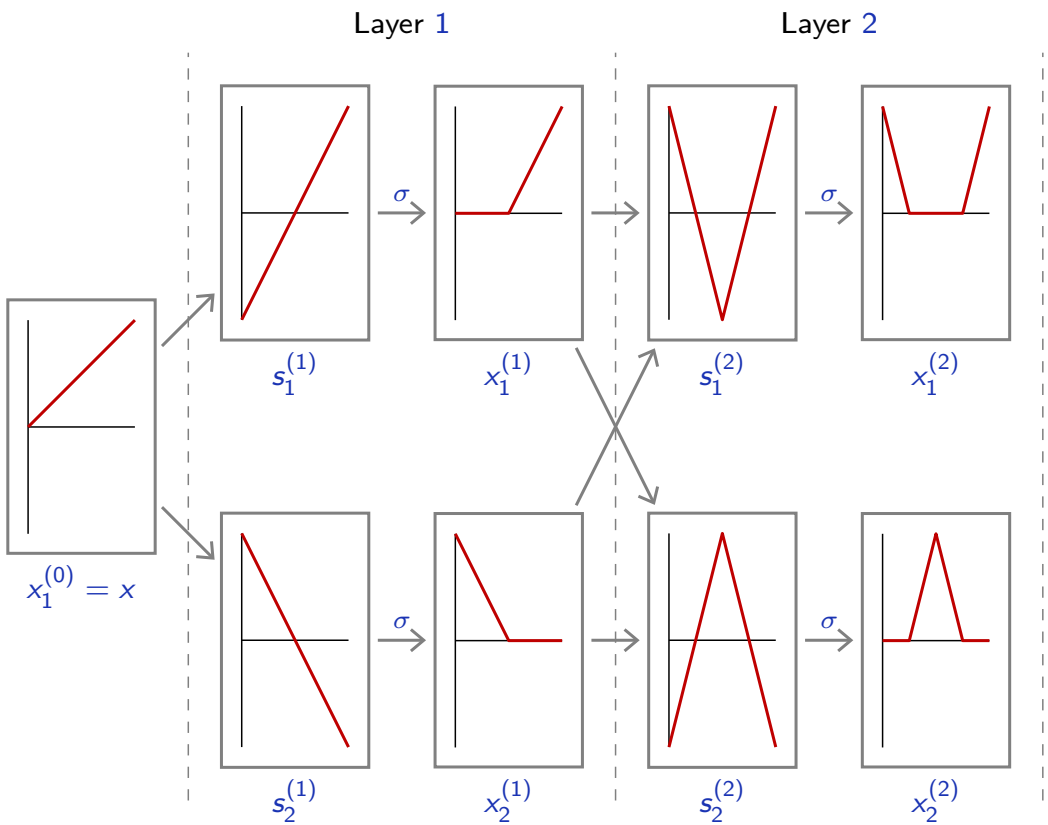

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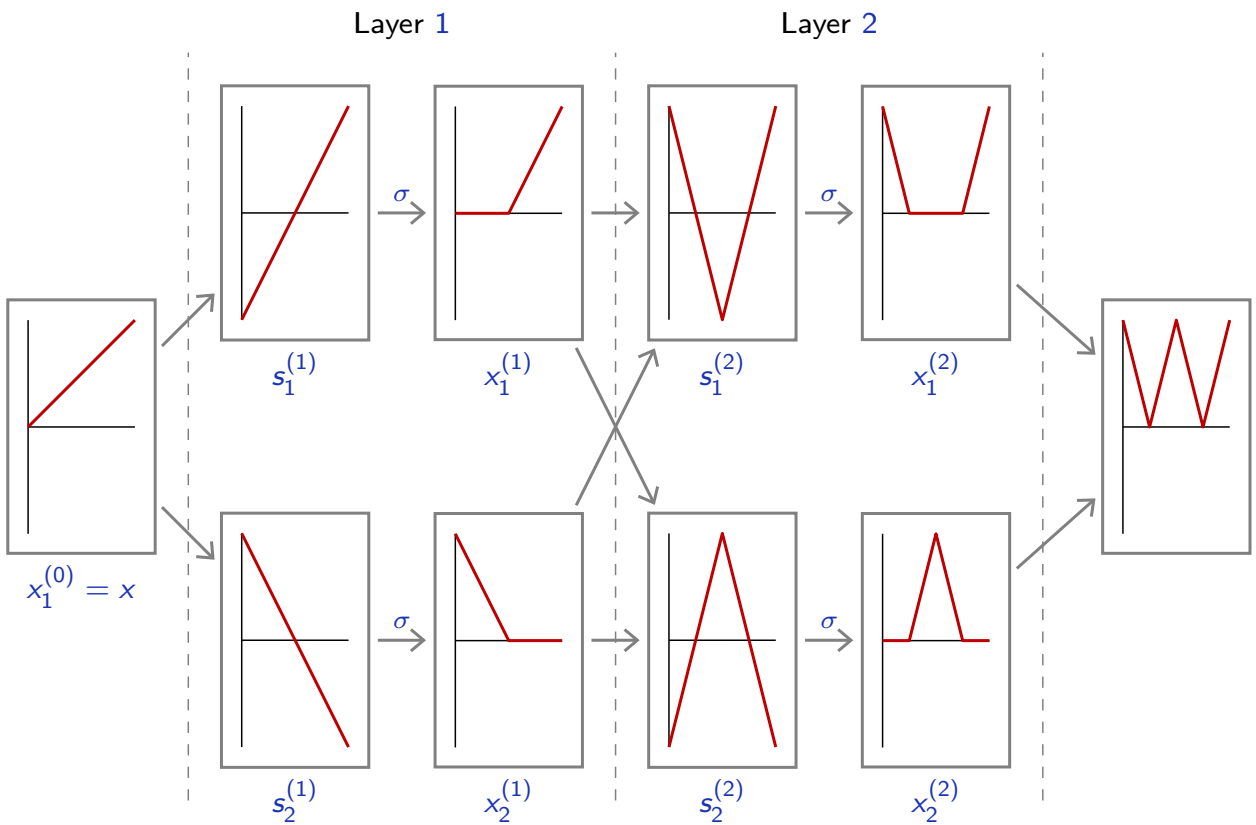

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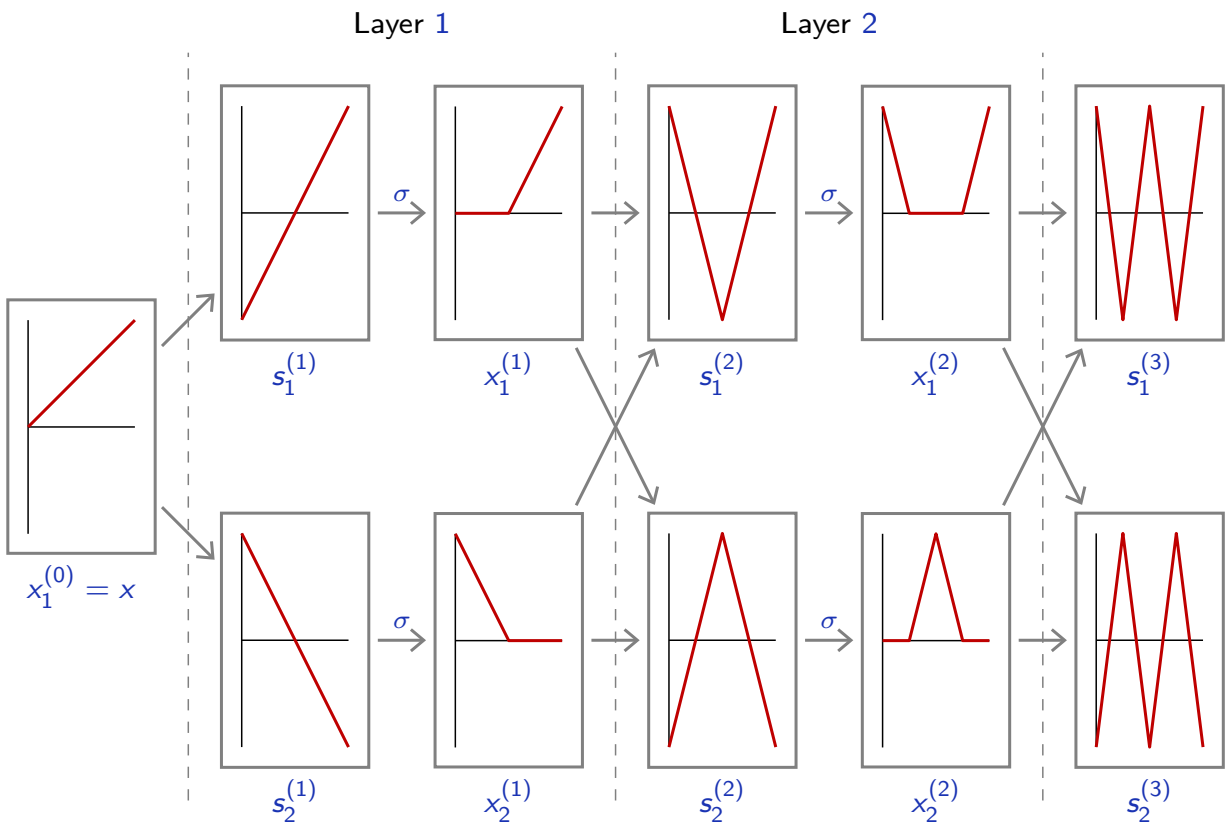

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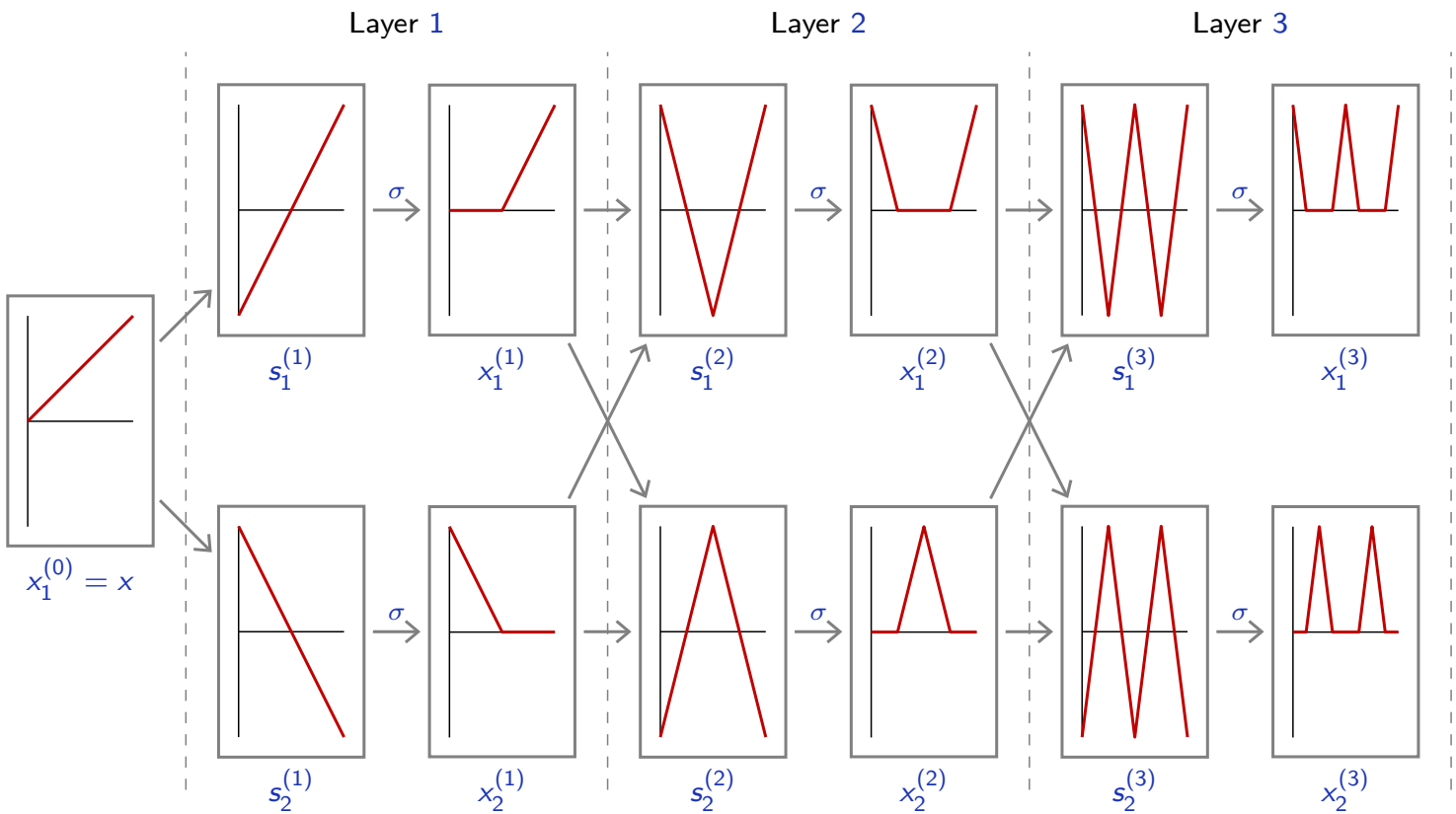

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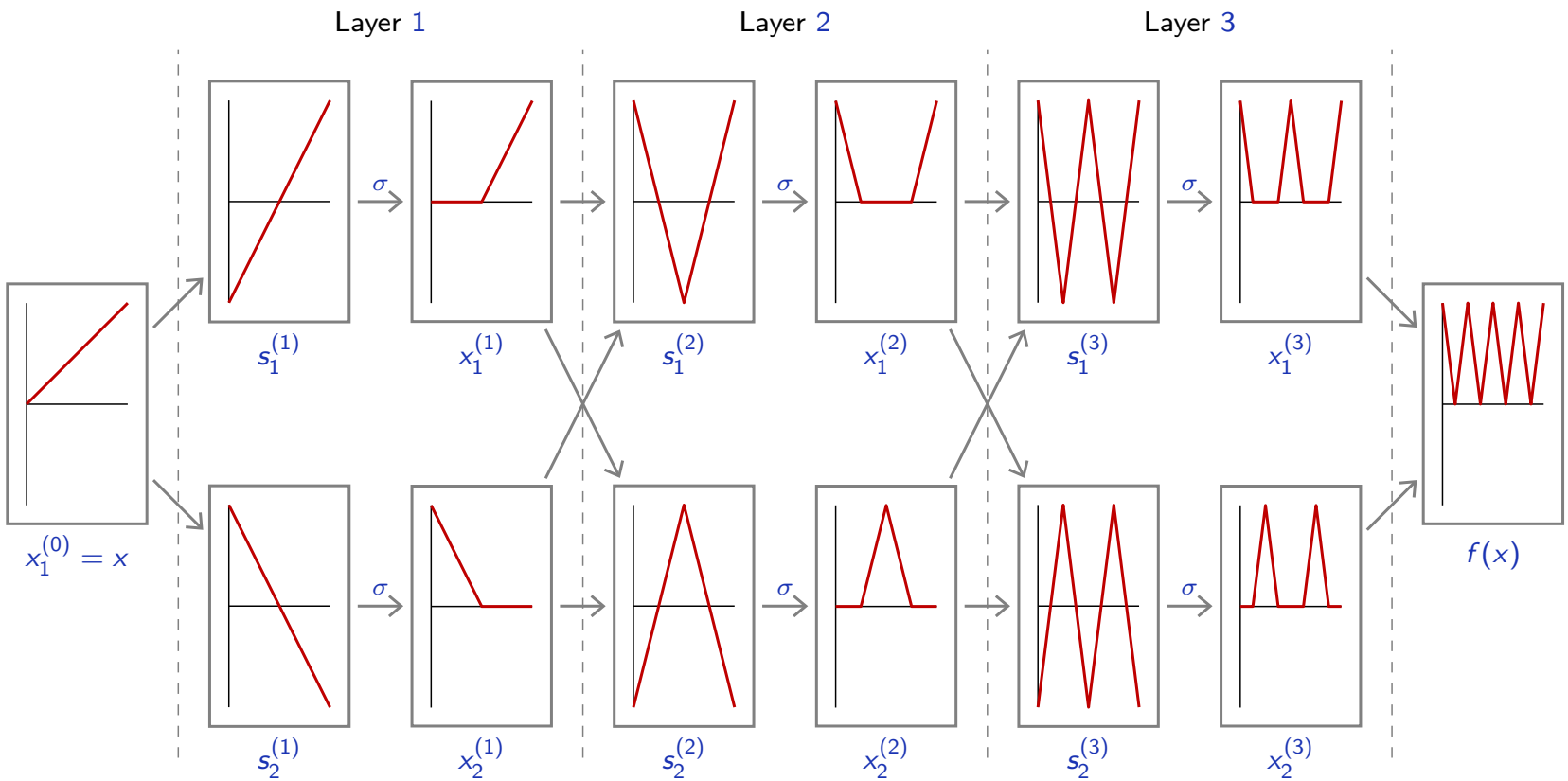
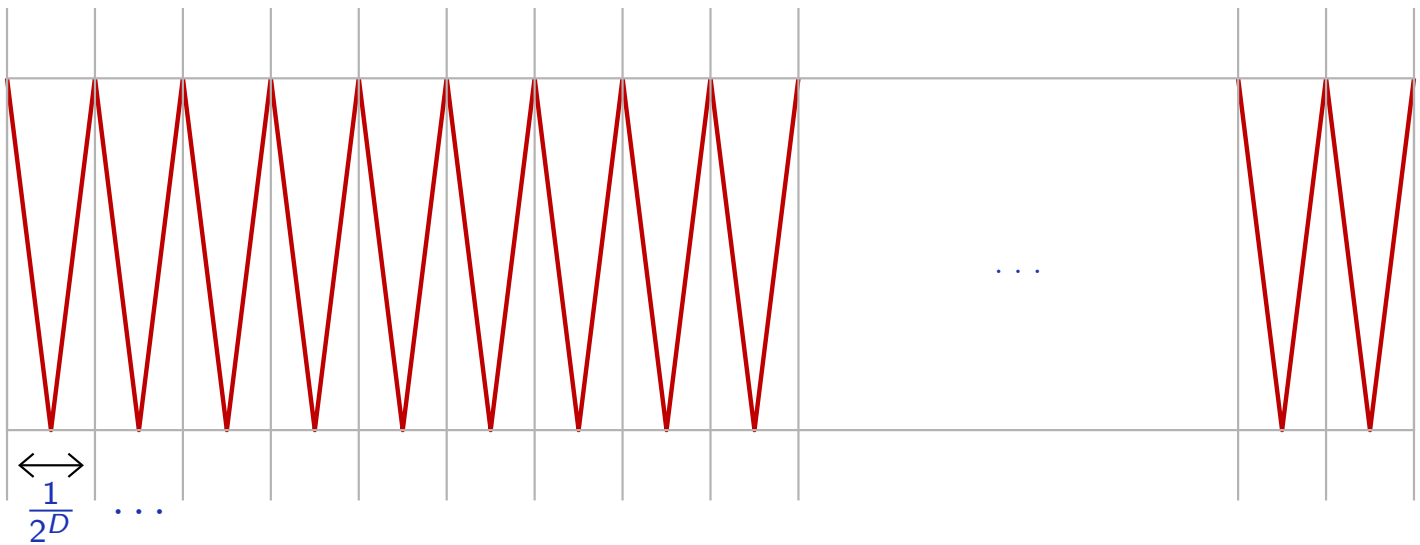
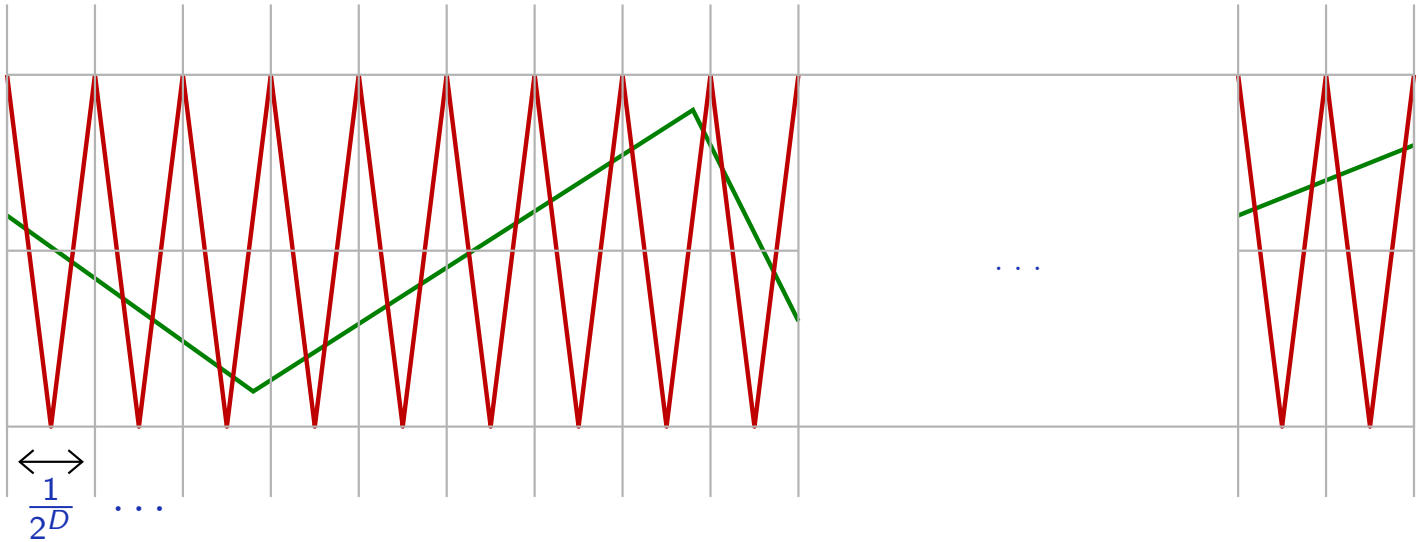


So for any D, there is a network with D hidden layers and $2D$ hidden units which computes an $f : [0, 1] \to [0, 1]$ of period $1/2^D$





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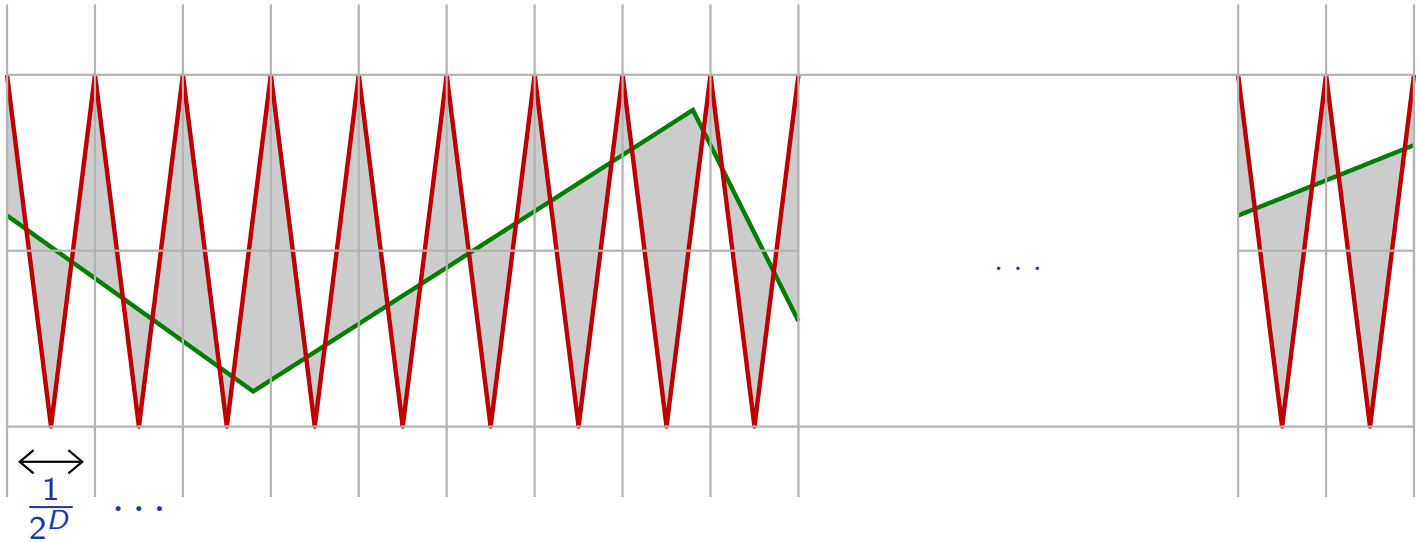
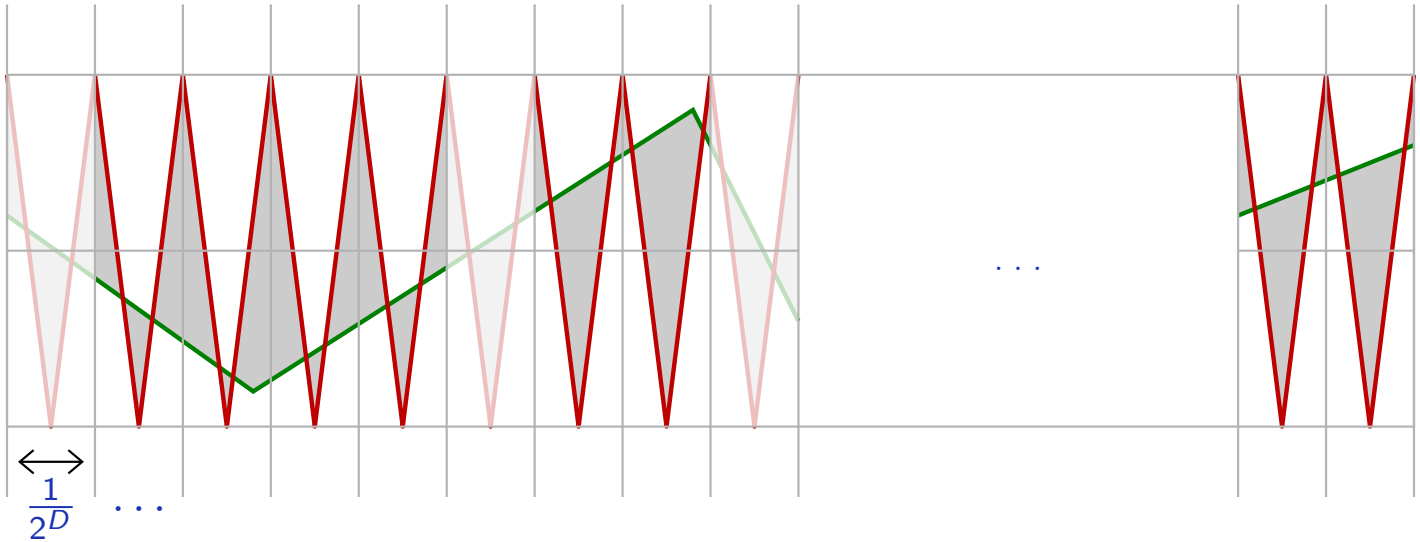


Given $g \in \mathcal{F}$

$$\int_0^1 |f(x) - g(x)|$$



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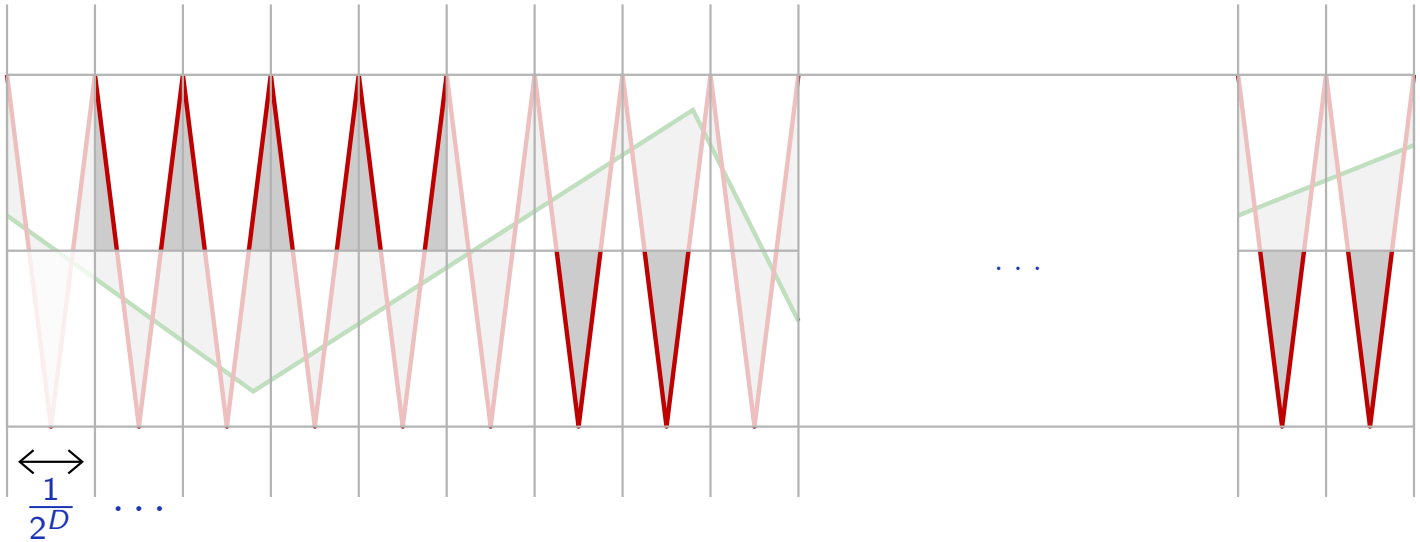

Given $g \in \mathscr{F}$, it crosses $\frac{1}{2}$ at most $\kappa(g)$ times, which means that on at least $2^D - \kappa(g)$ segments of length $1/2^D$, it is on one side of $\frac{1}{2}$, and

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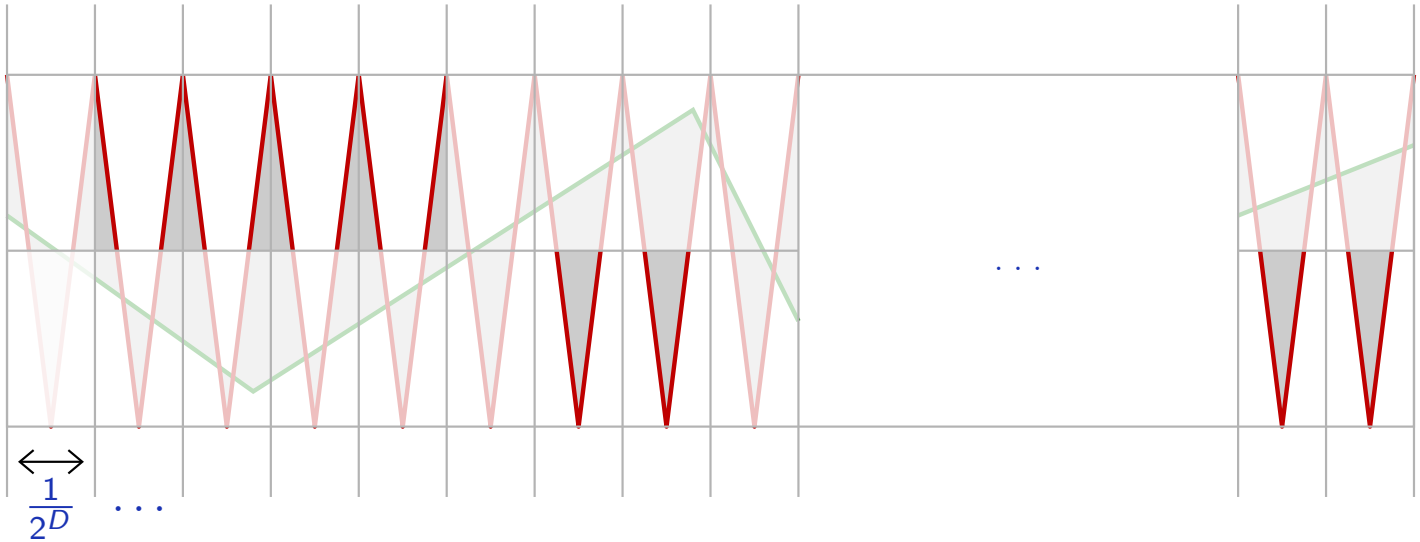

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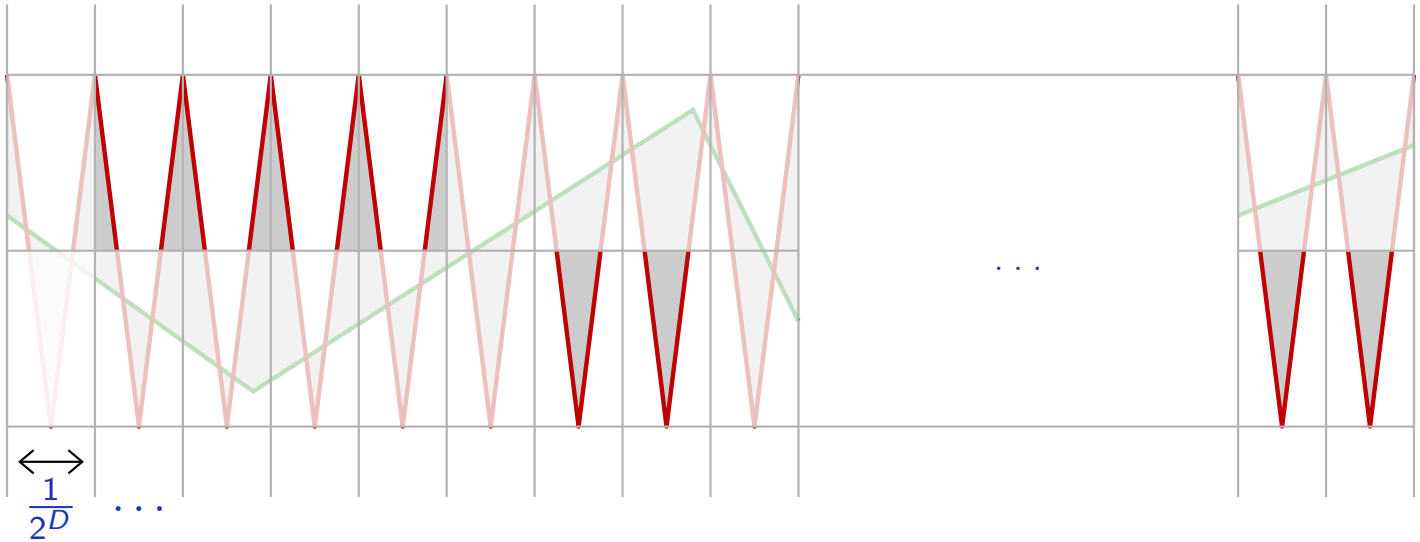
$$\begin{aligned}\int_0^1 |f(x) - g(x)| &\geq \left(2^D - \kappa(g)\right) \frac{1}{2} \int_0^{1/2^D} \left| f(x) - \frac{1}{2} \right| \\ &= \left(2^D - \kappa(g)\right) \frac{1}{2} \frac{1}{2^D} \frac{1}{8} \\ &= \frac{1}{16} \left(1 - \frac{\kappa(g)}{2^D}\right).\end{aligned}$$



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And we multiply f by 16 to get rid of the $\frac{1}{16}$.

So, considering ReLU MLPs with a single input/output:

There exists a network f with D^* layers, and $2D^*$ internal units, such that, for any network g with D layers of sizes $\{W^{(1)}, \ldots, W^{(D)}\}$:

$$\|f - g\|_1 \geq 1 - \frac{2^D}{2^{D^*}} \prod_{d=1}^{D} W^{(d)}.$$

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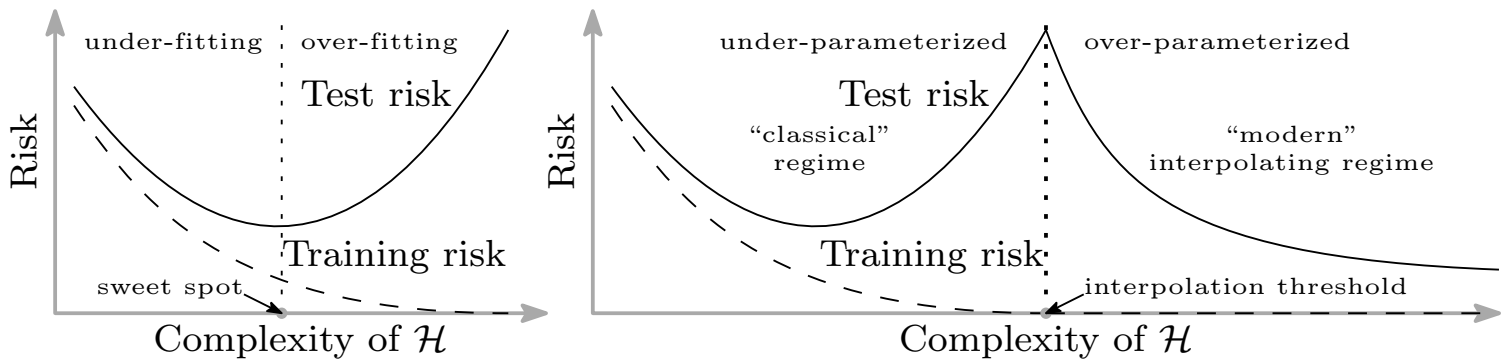
This is a simplified variant of results by Telgarsky (2015, 2016).

Regarding over-fitting, over-parametrizing a deep model often improves test performance, contrary to what the bias-variance decomposition predicts (Belkin et al., 2018).



(a) U-shaped "bias-variance" risk curve

(b) "double descent" risk curve

Figure 1: Curves for training risk (dashed line) and test risk (solid line). (a) The classical *U-shaped risk curve* arising from the bias-variance trade-off. (b) The *double descent risk curve*, which incorporates the U-shaped risk curve (i.e., the "classical" regime) together with the observed behavior from using high complexity function classes (i.e., the "modern" interpolating regime), separated by the interpolation threshold. The predictors to the right of the interpolation threshold have zero training risk.

(Belkin et al., 2018)

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In particular we have to ensure that

- the gradient does not "vanish" (Bengio et al., 1994; Hochreiter et al., 2001),
- gradient amplitude is homogeneous so that all parts of the network train at the same rate (Glorot and Bengio, 2010),
- the gradient does not vary too unpredictably when the weights change (Balduzzi et al., 2017).

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An additional issue for training very large architectures is the computational cost, which often turns out to be the main practical problem.

The end

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