Deep learning

5.4. L_2 and L_1 penalties

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https://fleuret.org/dlc/

Dec 20, 2020

We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

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If μ_W is a Gaussian density with a covariance matrix proportional to the identity, the log-prior $\log \mu_W(w)$ results in a quadratic penalty

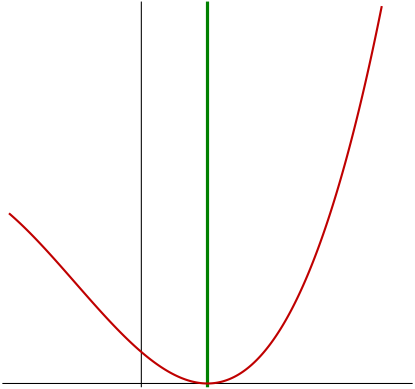
$$\lambda \|w\|_2^2.$$

Since this penalty is convex, its sum with a convex functional is convex.

This is called the L_2 regularization, or "weight decay" in the artificial neural network community.

Increasing the λ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

Since the derivative of $\|x\|_2^2$ is zero at zero, the optimal will never move there if it was not already there.



$$(x-1)^2 + \frac{1}{6}(x-1)^3$$

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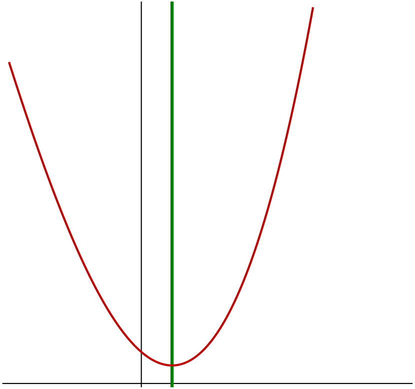

$$(x-1)^2 + \frac{1}{6}(x-1)^3 + x^2$$

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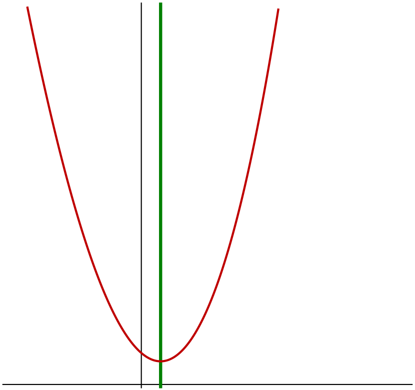

$$(x-1)^2 + \frac{1}{6}(x-1)^3 + 2x^2$$

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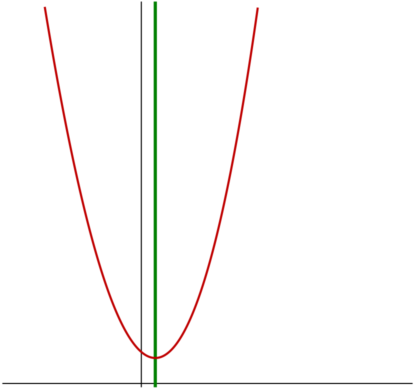

$$(x-1)^2 + \frac{1}{6}(x-1)^3 + 3x^2$$

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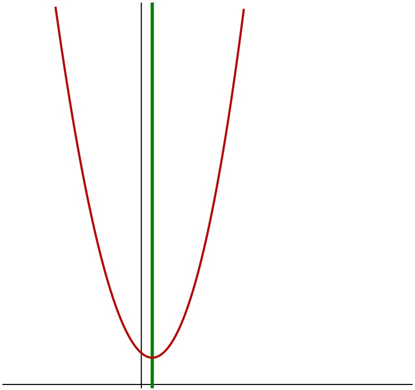


$$(x-1)^2 + \frac{1}{6}(x-1)^3 + 4x^2$$

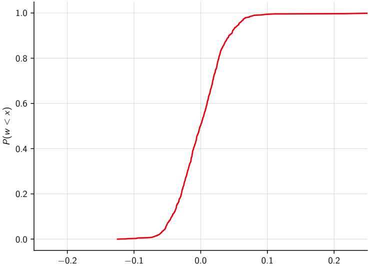
Convnet trained on MNIST with $1,000$ samples and a L_2 penalty.

	Error	
λ	Train	Test
0.000	0.000	0.064
0.001	0.000	0.063
0.002	0.000	0.064
0.004	0.005	0.065
0.010	0.022	0.075
0.020	0.048	0.101

```
output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])

for p in model.parameters():
    loss += lambda_l2 * p.pow(2).sum()

optimizer.zero_grad()
loss.backward()
optimizer.step()
```



$\lambda = 0.000$

Convnet trained on MNIST with $1,000$ samples and a L_2 penalty.

	Error	
λ	Train	Test
0.000	0.000	0.064
0.001	0.000	0.063
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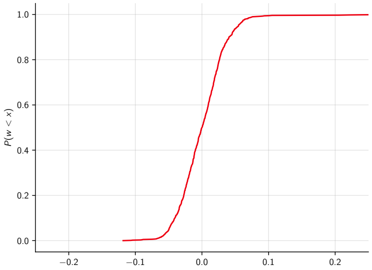

$\lambda = 0.001$

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0.000	0.000	0.064
0.001	0.000	0.063
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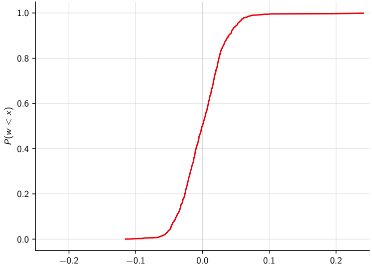

$\lambda = 0.002$

Convnet trained on MNIST with $1,000$ samples and a L_2 penalty.

	Error	
λ	Train	Test
0.000	0.000	0.064
0.001	0.000	0.063
0.002	0.000	0.064
0.004	0.005	0.065
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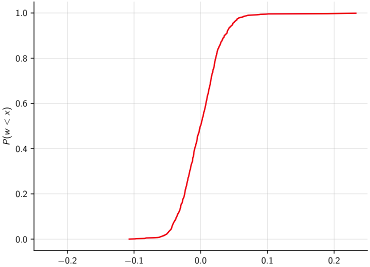

$\lambda = 0.004$

Convnet trained on MNIST with $1,000$ samples and a L_2 penalty.

	Error	
λ	Train	Test
0.000	0.000	0.064
0.001	0.000	0.063
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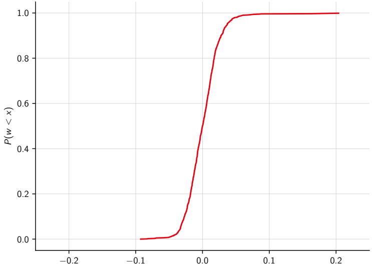

$\lambda = 0.010$

Convnet trained on MNIST with $1,000$ samples and a L_2 penalty.

	Error	
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0.000	0.000	0.064
0.001	0.000	0.063
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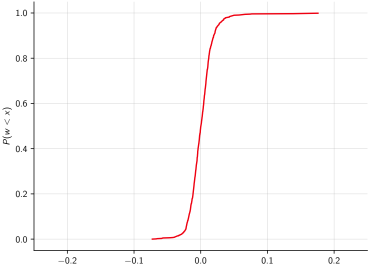

$\lambda = 0.020$

We can apply the exact same scheme with a Laplace prior

$$\mu(w) = \frac{1}{(2b)^D} \exp\left(-\frac{\|w\|_1}{b}\right)$$
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$$\lambda \|w\|_1.$$

This is the L_1 regularization.

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which results in a penalty term of the form

$$\lambda \|w\|_1.$$

This is the L_1 regularization. As for the L_2, this penalty is convex, and its sum with a convex functional is convex.

An important property of the L_1 penalty is that, if $\mathscr{L}$ is convex, and

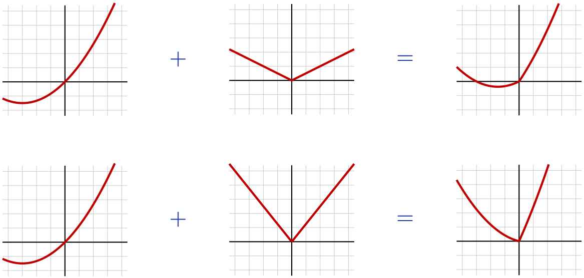
$$w^* = \underset{w}{\operatorname{argmin}}\, \mathscr{L}(w) + \lambda \|w\|_1$$

then

$$\forall d,\ \left| \frac{\partial \mathscr{L}}{\partial w_d}(w^*) \right| < \lambda \ \Rightarrow \ w_d^* = 0.$$

In practice it means that this penalty pushes some of the variables to zero, but contrary to the L_2 penalty they actually move and remain there.

The λ parameter controls the sparsity of the solution.



With the L_1 penalty, the update rule becomes

$$w_{t+1} = w_t - \eta \left(g_t + \lambda \operatorname{sign}(w_t) \right),$$

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$$\begin{aligned} w'_t &= w_t - \eta g_t \\ w_{t+1} &= w'_t - \eta \lambda \operatorname{sign}(w'_t). \end{aligned}$$

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This update may overshoot, and result in a component of w'_t strictly on one side of 0, while the same component in w_{t+1} is strictly on the other.

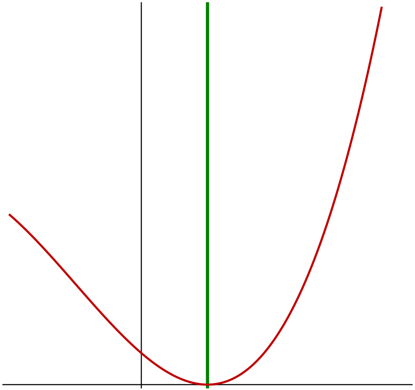
While this is not a problem in principle, since w_t will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (*e.g.* sparse coding to reduce memory footprint or computation).

The **proximal operator** prevents parameters from "crossing zero", by adapting λ when it is too large

$$
\begin{aligned}
w'_t &= w_t - \eta g_t \\
w_{t+1} &= w'_t - \eta \min(\lambda, |w'_t|) \odot \operatorname{sign}(w'_t).
\end{aligned}
$$

where $\min$ is component-wise, and $\odot$ is the Hadamard component-wise product.

Increasing the λ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.



$(x-1)^2 + \frac{1}{6}(x-1)^3$

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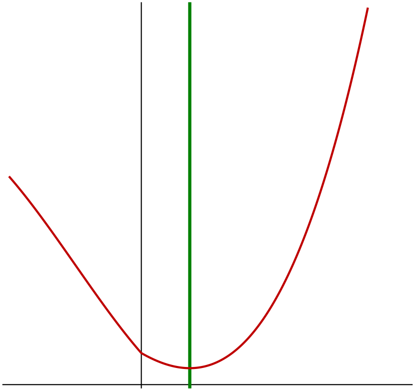

$$(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{2}|x|$$

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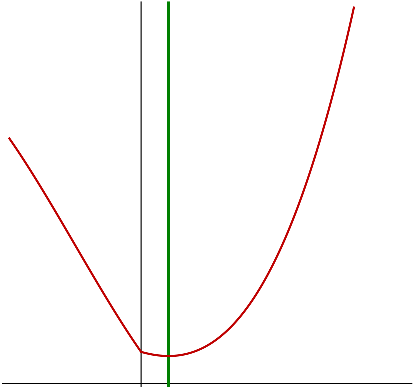

$$(x-1)^2 + \frac{1}{6}(x-1)^3 + |x|$$

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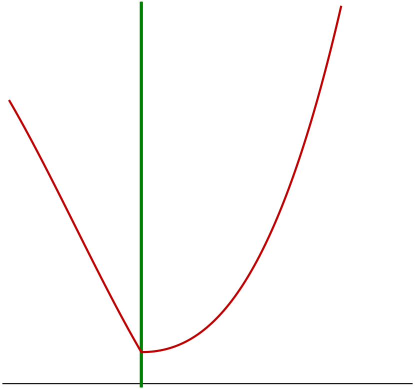

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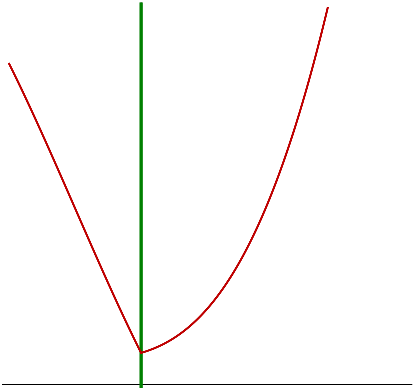

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Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
0.00010	0.087	0.128
0.00020	0.057	0.101
0.00050	0.496	0.516

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optimizer.step()

with torch.no_grad():
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        p.sub_(p.sign() * p.abs().clamp(max = lambda_l1))
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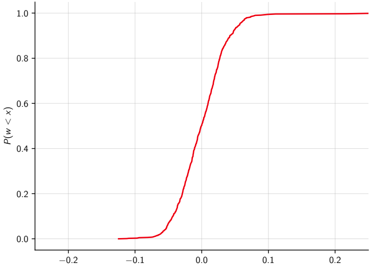

$\lambda = 0.00000$

Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
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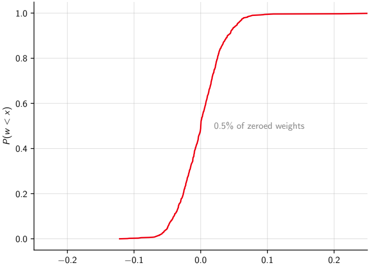

$\lambda = 0.00001$

Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
0.00010	0.087	0.128
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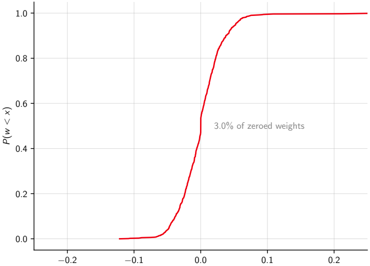

$\lambda = 0.00002$

Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
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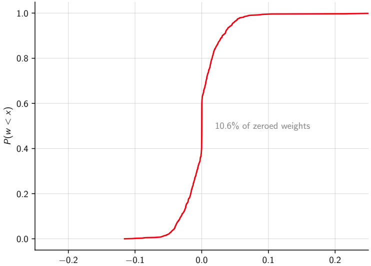

$\lambda = 0.00005$

Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
0.00010	0.087	0.128
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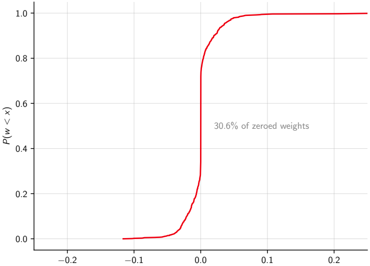

$\lambda = 0.0001$

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0.00000	0.000	0.064
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0.00002	0.000	0.067
0.00005	0.004	0.068
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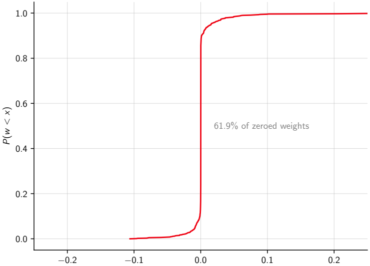

$\lambda = 0.0002$

Convnet trained on MNIST with $1,000$ samples and a L_1 penalty.

	Error	
λ	Train	Test
0.00000	0.000	0.064
0.00001	0.000	0.063
0.00002	0.000	0.067
0.00005	0.004	0.068
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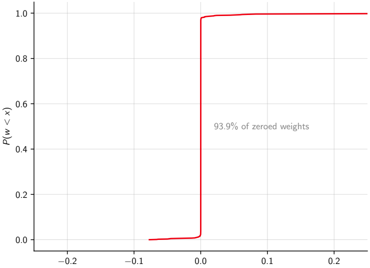

$\lambda = 0.0005$

Penalties on the weights may be useful when dealing with small models and small data-sets and are still standard when data is scarce.

While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines.

The end