Deep learning

5.4. $L_2$ and $L_1$ penalties

François Fleuret
https://fleuret.org/dlc/
Dec 20, 2020
We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

\[ \log \mu_W(w \mid \mathcal{D} = d) = \log \mu_{\mathcal{D}}(d \mid W = w) + \log \mu_W(w) - \log Z. \]
We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

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\log \mu_W(w \mid \mathcal{D} = d) = \log \mu_{\mathcal{D}}(d \mid W = w) + \log \mu_W(w) - \log Z.
\]

If \( \mu_W \) is a Gaussian density with a covariance matrix proportional to the identity, the log-prior \( \log \mu_W(w) \) results in a quadratic penalty

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\lambda \|w\|_2^2.
\]
We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model:

$$\log \mu_W(w \mid \mathcal{D} = d) = \log \mu_{\mathcal{D}}(d \mid W = w) + \log \mu_W(w) - \log Z.$$ 

If $\mu_W$ is a Gaussian density with a covariance matrix proportional to the identity, the log-prior $\log \mu_W(w)$ results in a quadratic penalty

$$\lambda \| w \|^2_2.$$ 

Since this penalty is convex, its sum with a convex functional is convex.

This is called the $L_2$ regularization, or “weight decay” in the artificial neural network community.
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

Since the derivative of $\|x\|_2^2$ is zero at zero, the optimal will never move there if it was not already there.

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3$$
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

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Since the derivative of $\|x\|_2^2$ is zero at zero, the optimal will never move there if it was not already there.

\[
(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 2x^2
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Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

Since the derivative of $||x||_2^2$ is zero at zero, the optimal will never move there if it was not already there.

\[(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 3x^2\]
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

Since the derivative of $\|x\|_2^2$ is zero at zero, the optimal will never move there if it was not already there.

\[(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 4x^2\]
Convnet trained on MNIST with 1,000 samples and a $L_2$ penalty.

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output = model(train_input[b:b+batch_size])

loss = criterion(output, train_target[b:b+batch_size])

for p in model.parameters():
    loss += lambda_l2 * p.pow(2).sum()

optimizer.zero_grad()

loss.backward()

optimizer.step()
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\[
\text{output} = \text{model}([\text{train\_input}[b:b+\text{batch\_size}])
\]
\[
\text{loss} = \text{criterion}([\text{output}, \text{train\_target}[b:b+\text{batch\_size}])
\]
\[
\text{for } p \text{ in model.parameters():}
\]
\[
\quad \text{loss} + = \lambda_{l2} \times p.pow(2).\text{sum()}
\]
\[
\text{optimizer.zero\_grad()}
\]
\[
\text{loss.backward()}
\]
\[
\text{optimizer.step()}
\]

$\lambda = 0.001$
Convnet trained on MNIST with 1,000 samples and a $L_2$ penalty.

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loss = criterion(output, train_target[b:b+batch_size])
for p in model.parameters():
    loss += lambda_l2 * p.pow(2).sum()
optimizer.zero_grad()
loss.backward()
optimizer.step()
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\[
\lambda = 0.002
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optimizer.zero_grad()
loss.backward()
optimizer.step()
```

\[ P(w < x) \]

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We can apply the exact same scheme with a Laplace prior

\[ \mu(w) = \frac{1}{(2b)^D} \exp \left( - \frac{\|w\|_1}{b} \right) \]

\[ = \frac{1}{(2b)^D} \exp \left( - \frac{1}{b} \sum_{d=1}^{D} |w_d| \right), \]
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which results in a penalty term of the form

$$\lambda \|w\|_1.$$ 

This is the $L_1$ regularization.
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which results in a penalty term of the form

\[
\lambda \|w\|_1.
\]

This is the $L_1$ regularization. As for the $L_2$, this penalty is convex, and its sum with a convex functional is convex.
An important property of the $L_1$ penalty is that, if $\mathcal{L}$ is convex, and

$$w^* = \arg\min_w \mathcal{L}(w) + \lambda \|w\|_1$$

then

$$\forall d, \left| \frac{\partial \mathcal{L}}{\partial w_d}(w^*) \right| < \lambda \implies w^*_d = 0.$$
In practice it means that this penalty pushes some of the variables to zero, but contrary to the $L_2$ penalty they actually move and remain there.

The $\lambda$ parameter controls the sparsity of the solution.
With the $L_1$ penalty, the update rule becomes

$$w_{t+1} = w_t - \eta (g_t + \lambda \text{sign}(w_t)),$$

This update may overshoot, and result in a component of $w'_t$ strictly on one side of 0, while the same component in $w_{t+1}$ is strictly on the other. While this is not a problem in principle, since $w_t$ will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).
With the $L_1$ penalty, the update rule becomes

$$w_{t+1} = w_t - \eta (g_t + \lambda \text{sign}(w_t)),$$

where \text{sign} is applied per-component. This is almost identical to

$$w'_t = w_t - \eta g_t$$

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While this is not a problem in principle, since $w_t$ will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).
The proximal operator prevents parameters from “crossing zero”, by adapting \( \lambda \) when it is too large

\[
\begin{align*}
    w'_t &= w_t - \eta g_t \\
    w_{t+1} &= w'_t - \eta \min(\lambda, |w'_t|) \odot \text{sign}(w'_t).
\end{align*}
\]

where \text{min} is component-wise, and \( \odot \) is the Hadamard component-wise product.
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

\[(x - 1)^2 + \frac{1}{6}(x - 1)^3\]
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + \frac{1}{2}|x|$$
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + |x|$$
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

$$(x - 1)^2 + \frac{1}{6}(x - 1)^3 + \frac{3}{2}|x|$$
Increasing the $\lambda$ parameter moves the optimal closer to 0, and away from the optimal for the loss without penalty.

\[(x - 1)^2 + \frac{1}{6}(x - 1)^3 + 2|x|\]
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loss = criterion(output, train_target[b:b+batch_size])

optimizer.zero_grad()
loss.backward()
optimizer.step()

with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_l1))

$P(w < x)$

$\lambda = 0.00000$
Convnet trained on MNIST with 1,000 samples and a $L_1$ penalty.

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with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_11))
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with torch.no_grad():
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        p.sub_(p.sign() * p.abs().clamp(max = lambda_l1))
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optimizer.zero_grad()
loss.backward()
optimizer.step()

with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_l1))
```

$\lambda = 0.00005$
Convnet trained on MNIST with 1,000 samples and a $L_1$ penalty.

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```python
output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])
```

```python
optimizer.zero_grad()
loss.backward()
optimizer.step()
```

```python
with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_11))
```

\[ \lambda = 0.0002 \]
Convnet trained on MNIST with 1,000 samples and a $L_1$ penalty.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Train</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.00</td>
<td>0.064</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.00</td>
<td>0.063</td>
</tr>
<tr>
<td>0.00002</td>
<td>0.00</td>
<td>0.067</td>
</tr>
<tr>
<td>0.00005</td>
<td>0.004</td>
<td>0.068</td>
</tr>
<tr>
<td>0.00010</td>
<td>0.087</td>
<td>0.128</td>
</tr>
<tr>
<td>0.00020</td>
<td>0.057</td>
<td>0.101</td>
</tr>
<tr>
<td>0.00050</td>
<td>0.496</td>
<td>0.516</td>
</tr>
</tbody>
</table>

```python
output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])
optimizer.zero_grad()
loss.backward()
optimizer.step()

with torch.no_grad():
    for p in model.parameters():
        p.sub_(p.sign() * p.abs().clamp(max = lambda_l1))
```

93.9% of zeroed weights

$\lambda = 0.0005$
Penalties on the weights may be useful when dealing with small models and small data-sets and are still standard when data is scarce.

While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines.
The end