Deep learning

3.3. Linear separability and feature design

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https://fleuret.org/dlc/
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A Perceptron model is shown with the formula:

\[ y = \sigma(x \cdot \Phi(w \cdot x + b)) \]
This is similar to the polynomial regression. If we have

$$\Phi : x \mapsto (1, x, x^2, \ldots, x^D)$$

and

$$\alpha = (\alpha_0, \ldots, \alpha_D)$$

then

$$\sum_{d=0}^{D} \alpha_d x^d = \alpha \cdot \Phi(x).$$
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\[ \sum_{d=0}^{D} \alpha_d x^d = \alpha \cdot \Phi(x). \]

By increasing \( D \), we can approximate any continuous real function on a compact space (Stone-Weierstrass theorem).

It means that we can make the capacity as high as we want.
We can apply the same to a more realistic binary classification problem: MNIST’s “8” vs. the other classes with a perceptron.

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![Graph showing the relationship between the number of features and error rate for training and testing]

- **Train error**
- **Test error**

<table>
<thead>
<tr>
<th>Nb. of features</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
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<tr>
<td>2</td>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
Remember the bias-variance tradeoff we saw in 2.3. “Bias-variance dilemma”

\[ \mathbb{E}((Y - y)^2) = (\mathbb{E}(Y) - y)^2 + \mathbb{V}(Y) \].

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In particular, good features should be invariant to perturbations of the signal known to keep the value to predict unchanged.
Votes (K=11)
Prediction (K=11)
Training points
Votes, radial feature (K=11)
Prediction, radial feature (K=11)
A classical example is the “Histogram of Oriented Gradient” descriptors (HOG), initially designed for person detection.

Roughly: divide the image in $8 \times 8$ blocks, compute in each the distribution of edge orientations over 9 bins.

Dalal and Triggs (2005) combined them with a SVM, and Dollár et al. (2009) extended them with other modalities into the “channel features”.

![Image of HOG representation](image)
Many methods (perceptron, SVM, $k$-means, PCA, etc.) only require to compute $\kappa(x, x') = \Phi(x) \cdot \Phi(x')$ for any $(x, x')$.

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This is the **kernel trick**, which we will not talk about in this course.
Training a model composed of manually engineered features and a parametric model such as logistic regression is now referred to as “shallow learning”.

The signal goes through a single processing trained from data.
The end
References
