Deep learning

3.2. Probabilistic view of a linear classifier

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https://fleuret.org/dlc/
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Consider the following class populations

\[ \forall y \in \{0, 1\}, x \in \mathbb{R}^D, \mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} \exp \left( -\frac{1}{2} (x - m_y)\Sigma^{-1}(x - m_y)^T \right). \]

That is, they are Gaussian with the same covariance matrix \( \Sigma \). This is the homoscedasticity assumption.
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Intuitively we can map data linearly to make all the covariance matrices identity, there the Bayesian separation is a plan, so it is also in the original space.
We have

\[ P(Y = 1 \mid X = x) \]
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It follows that, with

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It follows that, with

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we get

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P(Y = 1 \mid X = x) = \sigma \left( \log \frac{\mu_{X \mid Y = 1}(x)}{\mu_{X \mid Y = 0}(x)} + \log \frac{P(Y = 1)}{P(Y = 0)} \right).
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$$= \sigma \left( \log \mu_{X|Y=1}(x) - \log \mu_{X|Y=0}(x) + Z \right)$$
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$$+ \frac{1}{2} x\Sigma^{-1}x^T - m_0\Sigma^{-1}x^T + \frac{1}{2} m_0\Sigma^{-1}m_0^T + Z \right)$$
So with our Gaussians $\mu_{X|Y=x}$ of same $\Sigma$, we get

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= \sigma \left( \frac{-1}{2} (x - m_1)\Sigma^{-1}(x - m_1)^T + \frac{1}{2} (x - m_0)\Sigma^{-1}(x - m_0)^T + Z \right)
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\[
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+ \frac{1}{2} x\Sigma^{-1}x^T - m_0\Sigma^{-1}x^T + \frac{1}{2} m_0\Sigma^{-1}m_0^T + Z \right)
\]

\[
= \sigma \left( \frac{(m_1 - m_0)\Sigma^{-1}}{w}x^T + \frac{1}{2} \left( m_0\Sigma^{-1}m_0^T - m_1\Sigma^{-1}m_1^T \right) + Z \right)
\]

The homoscedasticity makes the second-order terms vanish.
So with our Gaussians $\mu_{X|Y=y}$ of same $\Sigma$, we get

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$$= \sigma \left( -\frac{1}{2} x \Sigma^{-1} x^T + m_1 \Sigma^{-1} x^T - \frac{1}{2} m_1 \Sigma^{-1} m_1^T \right.$$

$$+ \frac{1}{2} x \Sigma^{-1} x^T - m_0 \Sigma^{-1} x^T + \frac{1}{2} m_0 \Sigma^{-1} m_0^T + Z \bigg)$$

$$= \sigma \left( (m_1 - m_0) \Sigma^{-1} x^T + \frac{1}{2} \left( m_0 \Sigma^{-1} m_0^T - m_1 \Sigma^{-1} m_1^T \right) + Z \right)$$

$$= \sigma(w \cdot x + b).$$

The homoscedasticity makes the second-order terms vanish.
\mu_{X|Y=0} \quad \mu_{X|Y=1}

\begin{align*}
P(Y = 1 | X = x)
\end{align*}
$\mu_{X|Y=0}$

$\mu_{X|Y=1}$

$P(Y = 1 \mid X = x)$
\[ \mu_{X|Y=0} \quad \mu_{X|Y=1} \]

\[ P(Y = 1 | X = x) \]
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So the overall model

\[ f(x; w, b) = \sigma(w \cdot x + b) \]

looks very similar to the perceptron.
We can use the model from LDA

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but instead of modeling the densities and derive the values of \( w \) and \( b \), directly compute them by maximizing their probability given the training data.
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First, to simplify the next slide, note that we have

\[ 1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x), \]

hence if \( Y \) takes value in \( \{-1, 1\} \) then

\[ \forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)). \]
We have

$$\log \mu_{W,B}(w, b \mid \mathcal{D} = d)$$

$$= \log \frac{\mu_{\mathcal{D}}(d \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(d)}$$

$$= \log \mu_{\mathcal{D}}(d \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z$$

$$= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'$$
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= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'
$$

This is the \textbf{logistic regression}, whose loss aims at minimizing

$$
- \log \sigma(y_nf(x_n)).
$$
Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.
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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.
The end