Deep learning

3.2. Probabilistic view of a linear classifier

François Fleuret

https://fleuret.org/dlc/

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Consider the following class populations

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\[ \mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D|\Sigma|}} \exp \left( -\frac{1}{2} (x - m_y)\Sigma^{-1}(x - m_y)^T \right). \]

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Intuitively we can map data linearly to make all the covariance matrices identity, there the Bayesian separation is a plan, so it is also in the original space.
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So with our Gaussians $\mu_{X|Y=y}$ of same $\Sigma$, we get

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The homoscedasticity makes the second-order terms vanish.
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So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.
We can use the model from LDA

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but instead of modeling the densities and derive the values of \( w \) and \( b \), directly compute them by maximizing their probability given the training data.
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First, to simplify the next slide, note that we have

\[ 1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x), \]

hence if \( Y \) takes value in \( \{-1, 1\} \) then

\[ \forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)). \]
We have

\[ \log \mu_{W,B}(w, b \mid \mathcal{D} = d) = \log \frac{\mu_{\mathcal{D}}(d \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(d)} = \log \mu_{\mathcal{D}}(d \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z \]

\[ = \sum_{n} \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z' \]
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This is the **logistic regression**, whose loss aims at minimizing

$$- \log \sigma(y_n f(x_n)).$$
Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.
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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.
The end