Deep learning

3.2. Probabilistic view of a linear classifier

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Consider the following class populations

$$\forall y \in \{0,1\}, x \in \mathbb{R}^D,$$
$$\mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D|\Sigma|}} \exp\left(-\frac{1}{2}(x - m_y)\Sigma^{-1}(x - m_y)^T\right).$$

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Intuitively we can map data linearly to make all the covariance matrices identity, there the Bayesian separation is a plan, so it is also in the original space.

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P(Y = 1 \mid X = x) &= \frac{\mu_{X|Y=1}(x) P(Y = 1)}{\mu_X(x)} \\
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P(Y=1 \mid X=x) = \sigma\left(\log \frac{\mu_{X \mid Y=1}(x)}{\mu_{X \mid Y=0}(x)} + \log \frac{P(Y=1)}{P(Y=0)} \right).
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&= \sigma\left(\underbrace{(m_1 - m_0)\Sigma^{-1}}_{w} x^T + \underbrace{\frac{1}{2}\left(m_0\Sigma^{-1}m_0^T - m_1\Sigma^{-1}m_1^T\right) + Z}_{b}\right)
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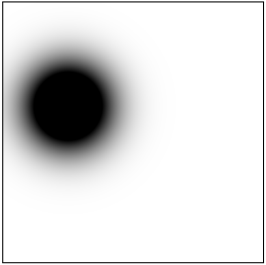
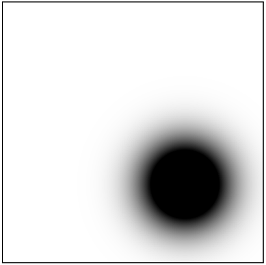
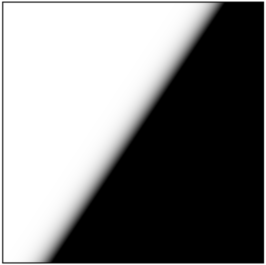
The homoscedasticity makes the second-order terms vanish.



$\mu_{X|Y=0}$



$\mu_{X|Y=1}$



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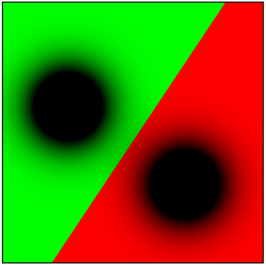

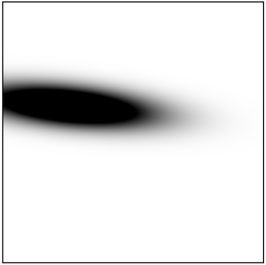

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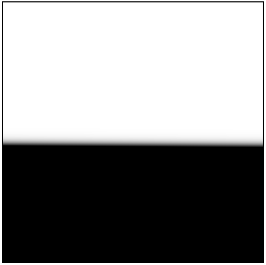

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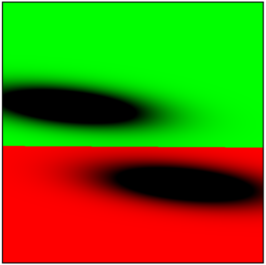

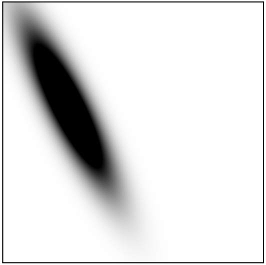

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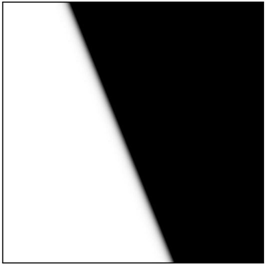

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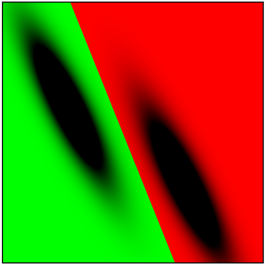

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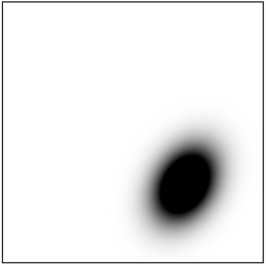
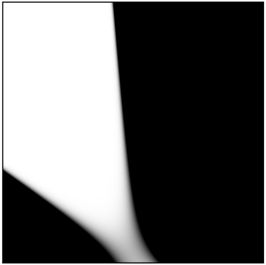


$\mu_{X|Y=1}$



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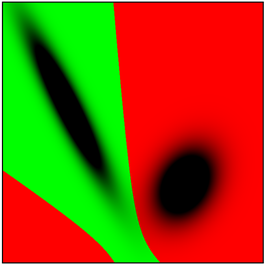
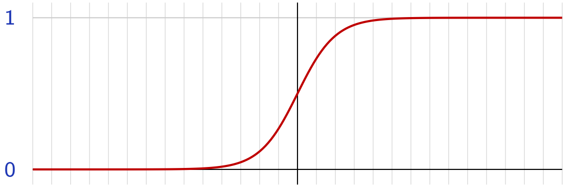


Note that the (logistic) sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}},$$

looks like a "soft heavyside"



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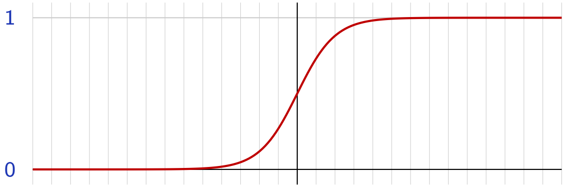

So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.

We can use the model from LDA

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First, to simplify the next slide, note that we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x),$$

hence if Y takes value in $\{-1, 1\}$ then

$$\forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)).$$

We have

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\begin{aligned}
&\log \mu_{W,B}(w, b \mid \mathscr{D} = \mathbf{d}) \\
&\qquad = \log \frac{\mu_{\mathscr{D}}(\mathbf{d} \mid W = w, B = b)\, \mu_{W,B}(w, b)}{\mu_{\mathscr{D}}(\mathbf{d})} \\
&\qquad = \log \mu_{\mathscr{D}}(\mathbf{d} \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z \\
&\qquad = \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'
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This is the **logistic regression**, whose loss aims at minimizing

$$- \log \sigma(y_n f(x_n)).$$

Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.

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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.

The end