Deep learning

3.1. The perceptron

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https://fleuret.org/ee559/

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The first mathematical model for a neuron was the Threshold Logic Unit, with Boolean inputs and outputs:

\[ f(x) = 1 \{ w \sum_i x_i + b \geq 0 \} \cdot \]

It can in particular implement \( \text{or} \) and \( \text{and} \) functions:

\[ \begin{align*}
\text{or} & : (u, v) = 1 \{ u + v - 0.5 \geq 0 \} \quad (w = 1, b = -0.5) \\
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\end{align*} \]

Not function:

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1 & \text{if } \sum_i w_i x_i + b \geq 0 \\
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It was originally motivated by biology, with \( w_i \) being the synaptic weights, and \( x_i \) and \( f \) firing rates. However, it is a (very) crude biological model.
To make things simpler we take responses \( \pm 1 \). Let

\[
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For neural networks, the function $\sigma$ that follows a linear operator is called the activation function.
We can represent this “neuron” as follows:
We can also use tensor operations, as in

\[ f(x) = \sigma(w \cdot x + b). \]
Given a training set

\[(x_n, y_n) \in \mathbb{R}^D \times \{-1, 1\}, \quad n = 1, \ldots, N,\]

a very simple scheme to train such a linear operator for classification is the \textbf{perceptron algorithm}:
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1. Start with \( w^0 = 0 \),
2. while \( \exists n_k \) s.t. \( y_{n_k} (w^k \cdot x_{n_k}) \leq 0 \), update \( w^{k+1} = w^k + y_{n_k} x_{n_k} \).
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The bias $b$ can be introduced as one of the $w$s by adding a constant component to $x$ equal to 1.
def train_perceptron(x, y, nb_epochs_max):
    w = torch.zeros(x.size(1))

    for e in range(nb_epochs_max):
        nb_changes = 0
        for i in range(x.size(0)):
            if x[i].dot(w) * y[i] <= 0:
                w = w + y[i] * x[i]
                nb_changes = nb_changes + 1
        if nb_changes == 0: break;

    return w
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epoch 0 nb_changes 64 train_error 0.23% test_error 0.19%
ePOCH 1 nb_changes 24 train_error 0.07% test_error 0.00%
ePOCH 2 nb_changes 10 train_error 0.06% test_error 0.05%
ePOCH 3 nb_changes 6 train_error 0.03% test_error 0.14%
ePOCH 4 nb_changes 5 train_error 0.03% test_error 0.09%
ePOCH 5 nb_changes 4 train_error 0.02% test_error 0.14%
ePOCH 6 nb_changes 3 train_error 0.01% test_error 0.14%
ePOCH 7 nb_changes 2 train_error 0.00% test_error 0.14%
ePOCH 8 nb_changes 0 train_error 0.00% test_error 0.14%
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We can get a convergence result under two assumptions:

1. The $x_n$ are in a sphere of radius $R$:
   $$\exists R > 0, \quad \forall n, \|x_n\| \leq R.$$  

2. The two populations can be separated with a margin $\gamma$:
   $$\exists w^*, \|w^*\| = 1, \quad \exists \gamma > 0, \quad \forall n, y_n(x_n \cdot w^*) \geq \gamma/2.$$
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To prove the convergence, let us make the assumption that there still is a misclassified sample at iteration $k$. We have

\[ w^{k+1} \cdot w^* = (w^k + y_n x_n) \cdot w^* \]
\[ = w^k \cdot w^* + y_n (x_n \cdot w^*) \]
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\end{align*}
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Since

\[
||w^k|| ||w^*|| \geq w^k \cdot w^*,
\]

we get

\[
||w^k||^2 \geq (w^k \cdot w^*)^2 / ||w^*||^2 \\
\geq k^2 \gamma^2 / 4.
\]
And

\[ \|w^{k+1}\|^2 = w^{k+1} \cdot w^{k+1} \]

\[ = (w^k + y_n x_n) \cdot (w^k + y_n x_n) \]

\[ = w^k \cdot w^k + 2y_n w^k \cdot x_n + \|x_n\|^2 \leq 0 \]

\[ \leq \|w^k\|^2 + R^2 \leq (k + 1) R^2. \]
Putting these two results together, we get

\[ \frac{k^2 \gamma^2}{4} \leq \|w^k\|^2 \leq k R^2 \]

hence

\[ k \leq \frac{4R^2}{\gamma^2}, \]

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This result makes sense:

- The bound does not change if the population is scaled, and
- the larger the margin, the more quickly the algorithm classifies all the samples correctly.
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Support Vector Machines (SVM) achieve this by minimizing

$$\mathcal{L}(w, b) = \lambda \|w\|^2 + \frac{1}{N} \sum_n \max(0, 1 - y_n(w \cdot x_n + b)),$$

which is convex and has a global optimum.
\[ \mathcal{L}(w, b) = \lambda \|w\|^2 + \frac{1}{N} \sum_n \max(0, 1 - y_n(w \cdot x_n + b)) \]

Support vectors

Minimizing \( \max(0, 1 - y_n(w \cdot x_n + b)) \) pushes the \( n \)th sample beyond the plane \( w \cdot x_n + b = y_n \), and minimizing \( \|w\|^2 \) increases the distance between the \( w \cdot x_n + b = \pm 1 \).

At convergence, only a small number of samples matter, the "support vectors".
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The term

\[ \max(0, 1 - \alpha) \]

is the so called “hinge loss”
The end
References
