The Linear Discriminant Analysis (LDA) algorithm provides a nice bridge between these linear classifiers and probabilistic modeling.

Consider the following class populations

\[
\forall y \in \{0, 1\}, x \in \mathbb{R}^D, \\
\mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D|\Sigma|}} \exp\left( -\frac{1}{2} (x - m_y)\Sigma^{-1}(x - m_y)^T \right). 
\]

That is, they are Gaussian with the same covariance matrix \( \Sigma \). This is the homoscedasticity assumption.

Intuitively we can map data linearly to make all the covariance matrices identity, there the Bayesian separation is a plan, so it is also in the original space.
We have

\[
P(Y = 1 \mid X = x) = \frac{\mu_{X \mid Y=1}(x)P(Y = 1)}{\mu_X(x)}
\]

\[
= \frac{\mu_{X \mid Y=1}(x)P(Y = 1)}{\mu_{X \mid Y=0}(x)P(Y = 0) + \mu_X(x)P(Y = 1)}
\]

\[
= \frac{1}{1 + \frac{\mu_{X \mid Y=0}(x)}{\mu_X(x)}P(Y = 0)}.
\]

It follows that, with

\[
\sigma(x) = \frac{1}{1 + e^{-x}},
\]

we get

\[
P(Y = 1 \mid X = x) = \sigma \left( \log \frac{\mu_{X \mid Y=1}(x)}{\mu_{X \mid Y=0}(x)} + \log \frac{P(Y = 1)}{P(Y = 0)} \right).
\]

So with our Gaussians \( \mu_{X \mid Y=\cdot} \) of same \( \Sigma \), we get

\[
P(Y = 1 \mid X = x)
\]

\[
= \sigma \left( \log \frac{\mu_{X \mid Y=1}(x)}{\mu_{X \mid Y=0}(x)} + \log \frac{P(Y = 1)}{P(Y = 0)} \right)
\]

\[
= \sigma \left( \log \mu_{X \mid Y=1}(x) - \log \mu_{X \mid Y=0}(x) + Z \right)
\]

\[
= \sigma \left( -\frac{1}{2} (x - m_1)\Sigma^{-1}(x - m_1)^T + \frac{1}{2} (x - m_0)\Sigma^{-1}(x - m_0)^T + Z \right)
\]

\[
= \sigma \left( -\frac{1}{2} x\Sigma^{-1}x^T + m_1\Sigma^{-1}x^T - \frac{1}{2} m_1\Sigma^{-1}m_1^T + \frac{1}{2} x\Sigma^{-1}x^T - m_0\Sigma^{-1}x^T + \frac{1}{2} m_0\Sigma^{-1}m_0^T + Z \right)
\]

\[
= \sigma \left( \frac{(m_1 - m_0)\Sigma^{-1}x^T + 1}{2} \left( m_0\Sigma^{-1}m_0^T - m_1\Sigma^{-1}m_1^T \right) + Z \right)
\]

\[
= \sigma(w \cdot x + b).
\]

The homoscedasticity makes the second-order terms vanish.
Note that the (logistic) sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}},$$

looks like a “soft heavyside”

So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.
We can use the model from LDA

\[ f(x; w, b) = \sigma(w \cdot x + b) \]

but instead of modeling the densities and derive the values of \( w \) and \( b \), directly compute them by maximizing their probability given the training data.

First, to simplify the next slide, note that we have

\[ 1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x), \]

hence if \( Y \) takes value in \( \{-1, 1\} \) then

\[ \forall y \in \{-1, 1\}, \quad P(Y = y | X = x) = \sigma(y(w \cdot x + b)). \]

We have

\[
\log \mu_{W, B}(w, b | D = d) = \log \frac{\mu_D(d | W = w, B = b) \mu_{W, B}(w, b)}{\mu_D(d)} = \log \mu_D(d | W = w, B = b) + \log \mu_{W, B}(w, b) - \log Z = \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W, B}(w, b) - \log Z'.
\]

This is the **logistic regression**, whose loss aims at minimizing

\[-\log \sigma(y_n f(x_n)).\]
Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.

We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.