5.2. Stochastic gradient descent

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So far, to minimize a loss of the form

\[ \mathcal{L}(w) = \sum_{n=1}^{N} \ell_n(f(x_n; w), y_n) \]

we have considered the gradient-descent algorithm, of the form

\[ w_{t+1} = w_t - \eta \nabla \mathcal{L}(w_t). \]
A straight-forward implementation would be

```python
for e in range(nb_epochs):
    output = model(train_input)
    loss = criterion(output, train_target)

    model.zero_grad()
    loss.backward()
    with torch.no_grad():
        for p in model.parameters():
            p -= eta * p.grad
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However, the memory footprint is proportional to the full set size. This can be mitigated by summing the gradient through “mini-batches”:

```python
for e in range(nb_epochs):
    model.zero_grad()

    for b in range(0, train_input.size(0), batch_size):
        output = model(train_input[b:b+batch_size])
        loss = criterion(output, train_target[b:b+batch_size])
        loss.backward()

    with torch.no_grad():
        for p in model.parameters(): p -= eta * p.grad
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While it makes sense in principle to compute the gradient exactly, in practice:

\[ \nabla \ell(w_t) = \sum_{n=1}^{N} \nabla \ell_n(w_t), \]

and when we compute \( \ell_n \), we have already computed \( \ell_1, \ldots, \ell_{n-1} \), and we could have a better estimate of \( w^* \) than \( w_t \).
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- It takes time to compute (more exactly all our time!).
- It is an empirical estimation of an hidden quantity, and any partial sum is also an unbiased estimate, although of greater variance.
- It is computed incrementally

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So instead of summing over all the samples and moving by $\eta$, we can visit only $M = N/K$ samples and move by $K \eta$, which would cut the computation by $K$.

Although this is an ideal case, there is redundancy in practice that results in similar behaviors.
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The **mini-batch stochastic gradient descent** is the standard procedure for deep learning. It consists of visiting the samples in “mini-batches”, each of a few tens of samples, and updating the parameters each time.

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**The stochastic behavior of this procedure helps evade local minima.**
So our exact gradient descent with mini-batches

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can be modified into the mini-batch stochastic gradient descent as follows:

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Limitation of the gradient descent
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Deep-learning generally relies on a smarter use of the gradient, using statistics over its past values to make a “smarter step” with the current one.
Momentum and moment estimation
The “vanilla” mini-batch stochastic gradient descent (SGD) consists of

\[ w_{t+1} = w_t - \eta g_t, \]

where

\[ g_t = \sum_{b=1}^{B} \nabla \ell_{n(t,b)}(w_t) \]

is the gradient summed over a mini-batch.
The first improvement is the use of a “momentum” to add inertia in the choice of the step direction

\[ u_t = \gamma u_{t-1} + \eta g_t \]
\[ w_{t+1} = w_t - u_t. \]

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  \[ (u = \gamma u + \eta g) \Rightarrow \left( u = \frac{\eta}{1 - \gamma} g \right), \]
- it dampens oscillations in narrow valleys.
\[ \eta = 5.0e^{-2}, \gamma = 0 \]
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The update rule is, **on each coordinate separately**

\[
\begin{align*}
    m_t &= \beta_1 m_{t-1} + (1 - \beta_1) g_t \\
    \hat{m}_t &= \frac{m_t}{1 - \beta_1} \\
    v_t &= \beta_2 v_{t-1} + (1 - \beta_2) g_t^2 \\
    \hat{v}_t &= \frac{v_t}{1 - \beta_2} \\
    w_{t+1} &= w_t - \frac{\eta}{\sqrt{\hat{v}_t} + \epsilon} \hat{m}_t
\end{align*}
\]

(Kingma and Ba, 2014)

This can be seen as a combination of momentum, with \( \hat{m}_t \), and a per-coordinate re-scaling with \( \hat{v}_t \).
\( \eta = 5.0 \times 10^{-2} \)
Adam, $\beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 1e^{-8}, \eta = 1.0e^{-1}$
These two core strategies have been used in multiple incarnations:

- Nesterov’s accelerated gradient,
- Adagrad,
- Adadelta,
- RMSprop,
- AdaMax,
- Nadam ...
The end
References
