AMMI – Introduction to Deep Learning

3.2. Probabilistic view of a linear classifier

François Fleuret
https://fleuret.org/ammi-2018/
Sat Oct 27 16:46:03 CAT 2018
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Consider the following class populations

\[ \forall y \in \{0, 1\}, \ x \in \mathbb{R}^D, \]

\[ \mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D|\Sigma|}} \exp \left( -\frac{1}{2} (x - m_y)^T \Sigma^{-1} (x - m_y) \right). \]

That is, they are Gaussian with the same covariance matrix \( \Sigma \). This is the homoscedasticity assumption.
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+ \frac{1}{2}x\Sigma^{-1}x^T - m_0\Sigma^{-1}x^T + \frac{1}{2}m_0\Sigma^{-1}m_0^T + a \right)$$

$$= \sigma \left( (m_1 - m_0)\Sigma^{-1}x^T + \frac{1}{2} \left( m_0\Sigma^{-1}m_0^T - m_1\Sigma^{-1}m_1^T \right) + a \right)$$

The homoscedasticity makes the second-order terms vanish.
So with our Gaussians $\mu_{X|Y=y}$ of same $\Sigma$, we get

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= \sigma(w \cdot x + b).
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So with our Gaussians $\mu_{X|Y=y}$ of same $\Sigma$, we get

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$\mu_{X|Y=0}$

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$P(Y = 1 | X = x)$
\[ \mu X \mid Y = 0 \]

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So the overall model

\[ f(x; w, b) = \sigma(w \cdot x + b) \]

looks very similar to the perceptron.
We can use the model from LDA

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but instead of modeling the densities and derive the values of \( w \) and \( b \), directly compute them by maximizing their probability given the training data.
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First, to simplify the next slide, note that we have

\[ 1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x), \]

hence if \( Y \) takes value in \( \{-1, 1\} \) then

\[ \forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)). \]
We have

\[
\log \mu_{W,B}(w, b \mid \mathcal{D} = \mathbf{d}) = \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(\mathbf{d})} = \log \mu_{\mathcal{D}}(\mathbf{d} \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z = \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'
\]
We have

$$\log \mu_{W,B}(w, b \mid D = d)$$

$$= \log \frac{\mu_{\mathcal{D}}(d \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(d)}$$

$$= \log \mu_{\mathcal{D}}(d \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z$$

$$= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'$$

This is the **logistic regression**, whose loss aims at minimizing

$$-\log \sigma(y_n f(x_n)).$$
Although the probabilistic and Bayesian formulations may be helpful in certain contexts, the bulk of deep learning is disconnected from such modeling.
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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.
The end