3.1. The perceptron

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The first mathematical model for a neuron was the Threshold Logic Unit, with Boolean inputs and outputs:

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It can in particular implement

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\begin{align*}
    or(u, v) &= 1\{u + v - 0.5 \geq 0\} & (w = 1, b = -0.5) \\
    and(u, v) &= 1\{u + v - 1.5 \geq 0\} & (w = 1, b = -1.5) \\
    not(u) &= 1\{- u + 0.5 \geq 0\} & (w = -1, b = 0.5)
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Hence, any Boolean function can be build with such units.

(McCulloch and Pitts, 1943)
The perceptron is very similar

\[ f(x) = \begin{cases} 1 & \text{if } \sum_i w_i x_i + b \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

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This model was originally motivated by biology, with \( w_i \) being the \textit{synaptic weights}, and \( x_i \) and \( f \) firing rates.

It is a (very) crude biological model.

(Rosenblatt, 1957)
To make things simpler we take responses \( \pm 1 \). Let

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The perceptron classification rule boils down to

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For neural networks, the function \( \sigma \) that follows a linear operator is called the activation function.
We can represent this “neuron” as follows:
We can also use tensor operations, as in

\[ f(x) = \sigma(w \cdot x + b). \]
Given a training set

\[(x_n, y_n) \in \mathbb{R}^D \times \{-1, 1\}, \quad n = 1, \ldots, N,\]

a very simple scheme to train such a linear operator for classification is the **perceptron algorithm:**
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1. Start with \(w^0 = 0\),
2. while \(\exists n_k\) s.t. \(y_{n_k} (w^k \cdot x_{n_k}) \leq 0\), update \(w^{k+1} = w^k + y_{n_k} x_{n_k}\).
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The bias \(b\) can be introduced as one of the \(w\)s by adding a constant component to \(x\) equal to 1.

(Rosenblatt, 1957)
def train_perceptron(x, y, nb_epochs_max):
    w = torch.zeros(x.size(1))

    for e in range(nb_epochs_max):
        nb_changes = 0
        for i in range(x.size(0)):
            if x[i].dot(w) * y[i] <= 0:
                w = w + y[i] * x[i]
                nb_changes = nb_changes + 1
        if nb_changes == 0: break;

    return w
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```
epoch 0 nb_changes 64 train_error 0.23% test_error 0.19%
epoch 1 nb_changes 24 train_error 0.07% test_error 0.00%
epoch 2 nb_changes 10 train_error 0.06% test_error 0.05%
epoch 3 nb_changes 6 train_error 0.03% test_error 0.14%
epoch 4 nb_changes 5 train_error 0.03% test_error 0.09%
epoch 5 nb_changes 4 train_error 0.02% test_error 0.14%
epoch 6 nb_changes 3 train_error 0.01% test_error 0.14%
epoch 7 nb_changes 2 train_error 0.00% test_error 0.14%
epoch 8 nb_changes 0 train_error 0.00% test_error 0.14%
```
This crude algorithm works often surprisingly well. With MNIST’s “0”s as negative class, and “1”s as positive one.
We can get a convergence result under two assumptions:

1. The $x_n$ are in a sphere of radius $R$:
   $$\exists R > 0, \forall n, \|x_n\| \leq R.$$

2. The two populations can be separated with a margin $\gamma > 0$.
   $$\exists w^*, \|w^*\| = 1, \exists \gamma > 0, \forall n, y_n(x_n \cdot w^*) \geq \gamma/2.$$
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To prove the convergence, let us make the assumption that there still is a misclassified sample at iteration $k$, and $w^{k+1}$ is the weight vector updated with it. We have

$$w^{k+1} \cdot w^* = \left( w^k + y_n x_n \right) \cdot w^*$$

$$= w^k \cdot w^* + y_n (x_n \cdot w^*)$$

$$\geq w^k \cdot w^* + \gamma / 2$$

$$\geq (k + 1) \gamma / 2.$$
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$$w^{k+1} \cdot w^* = (w^k + y_{n_k}x_{n_k}) \cdot w^* = w^k \cdot w^* + y_{n_k}(x_{n_k} \cdot w^*) \geq w^k \cdot w^* + \gamma/2 \geq (k + 1)\gamma/2.$$ 

Since

$$\|w^k\|\|w^*\| \geq w^k \cdot w^*,$$

we get

$$\|w^k\|^2 \geq (w^k \cdot w^*)^2 /\|w^*\|^2 \geq k^2\gamma^2 /4.$$
And

\[ \|w^{k+1}\|^2 = w^{k+1} \cdot w^{k+1} \]
\[ = (w^k + y_n x_n) \cdot (w^k + y_n x_n) \]
\[ = w^k \cdot w^k + 2 y_n w^k \cdot x_n + \|x_n\|^2 \]
\[ \leq 0 \]
\[ \leq R^2 \]
\[ \leq \|w^k\|^2 + R^2 \]
\[ \leq (k + 1) R^2. \]
Putting these two results together, we get

\[ \frac{k^2 \gamma^2}{4} \leq \|w^k\|^2 \leq k R^2 \]

hence

\[ k \leq \frac{4R^2}{\gamma^2}, \]

hence no misclassified sample can remain after \( \left\lfloor \frac{4R^2}{\gamma^2} \right\rfloor \) iterations.
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This result makes sense:

- The bound does not change if the population is scaled, and
- the larger the margin, the more quickly the algorithm classifies all the samples correctly.
The perceptron stops as soon as it finds a separating boundary.

Other algorithms maximize the distance of samples to the decision boundary, which improves robustness to noise.
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Support Vector Machines (SVM) achieve this by minimizing

$$\mathcal{L}(w, b) = \lambda \|w\|^2 + \frac{1}{N} \sum_n \max(0, 1 - y_n(w \cdot x_n + b)),$$

which is convex and has a global optimum.
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Minimizing \( \max(0, 1 - y_n(w \cdot x_n + b)) \) pushes the \( n \)th sample beyond the plane \( w \cdot x + b = y_n \), and minimizing \( \|w\|^2 \) increases the distance between the \( w \cdot x + b = \pm 1 \).
\[ \mathcal{L}(w, b) = \lambda \|w\|^2 + \frac{1}{N} \sum_n \max(0, 1 - y_n(w \cdot x_n + b)) \]

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At convergence, only a small number of samples matter, the “support vectors”.
The term \[ \max(0, 1 - \alpha) \]
is the so called “hinge loss”
The end
References
