EE-559 – Deep learning

4.4. Convolutions

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https://fleuret.org/ee559/

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If they were handled as normal "unstructured" vectors, large-dimension signals such as sound samples or images would require models of intractable size.

For instance a linear layer taking a 256×256 RGB image as input, and producing an image of same size would require

$$(256 \times 256 \times 3)^2 \simeq 3.87e{+}10$$

parameters, with the corresponding memory footprint ($\simeq$150Gb !), and excess of capacity.

Moreover, this requirement is inconsistent with the intuition that such large signals have some “invariance in translation”. **A representation meaningful at a certain location can / should be used everywhere.**

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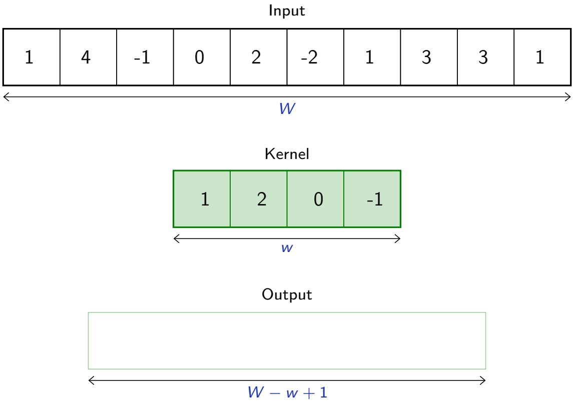
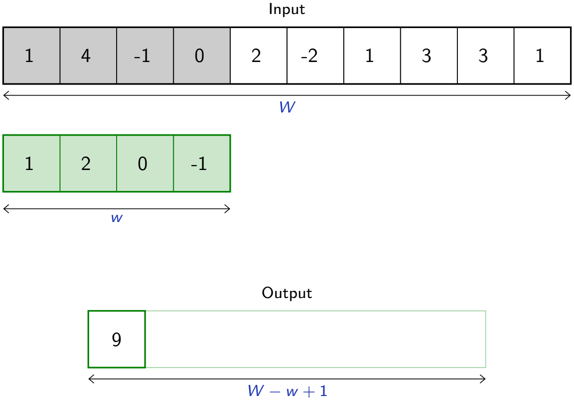
A convolution layer embodies this idea. It applies the same linear transformation locally, everywhere, and preserves the signal structure.

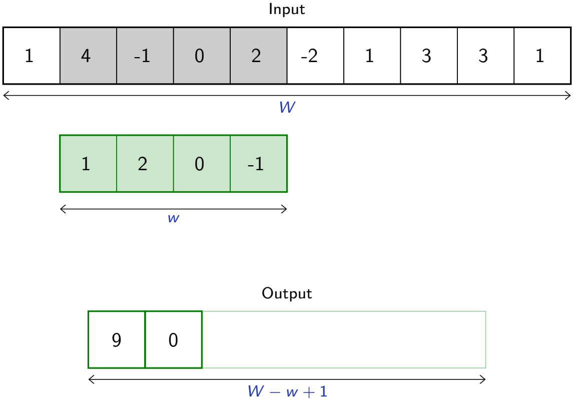
Input

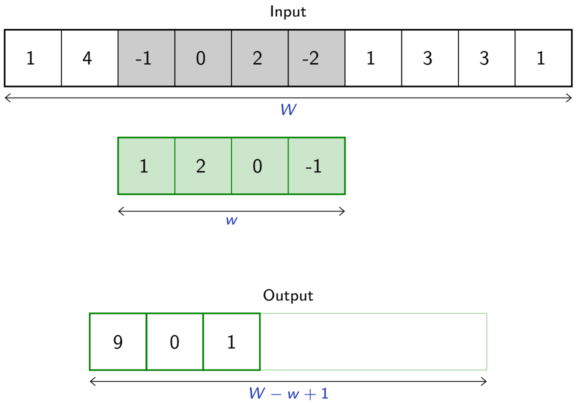
1	4	-1	0	2	-2	1	3	3	1

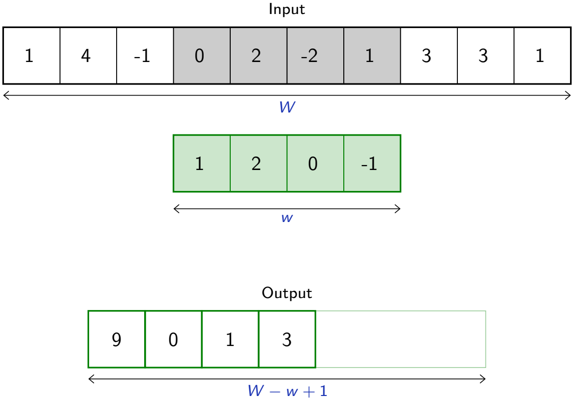
W

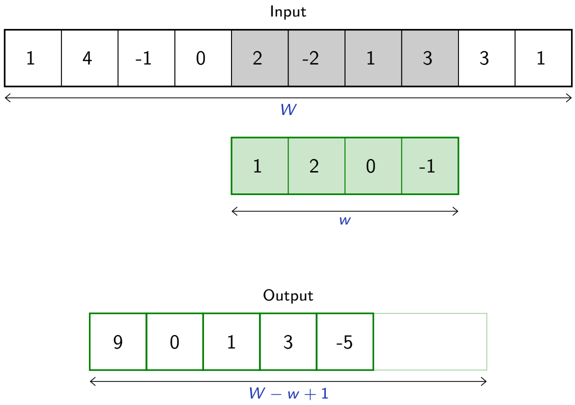

Input
1 4 -1 0 2 -2 1 3 3 1
W
Kernel
1 2 0 -1
w
Output
$W - w + 1$

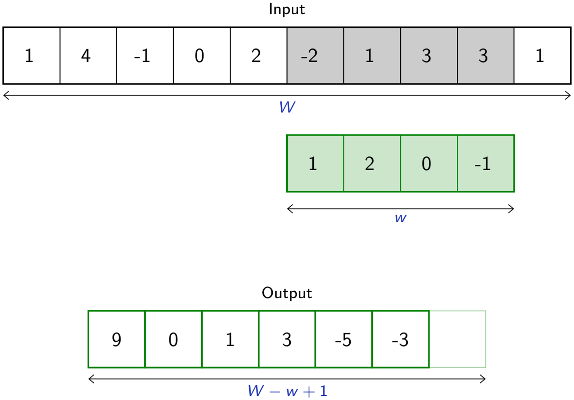

Input
1
4
-1
0
2
-2
1
3
3
1
W
1
2
0
-1
w
Output
9
W − w + 1

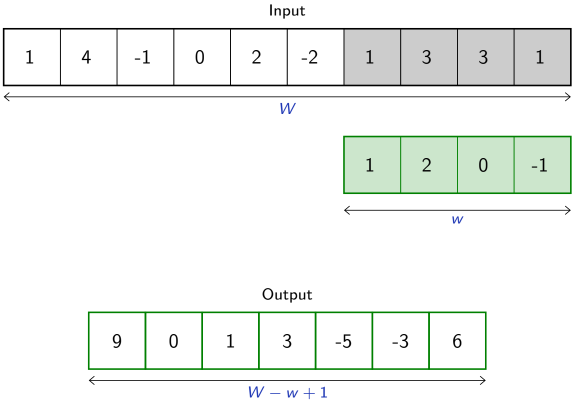

Input
1
4
-1
0
2
-2
1
3
3
1
W
1
2
0
-1
w
Output
9
0
W − w + 1

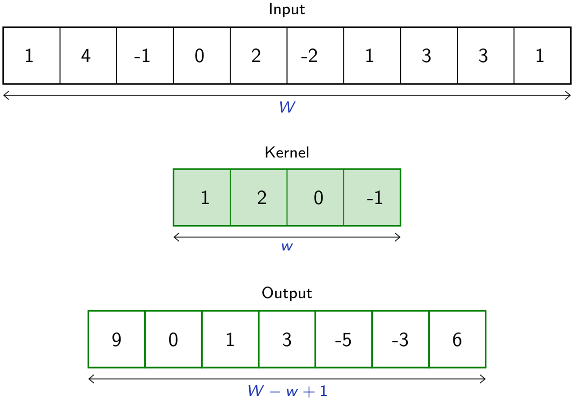

Input
1
4
-1
0
2
-2
1
3
3
1
W
1
2
0
-1
w
Output
9
0
1
$W - w + 1$


Input
1
4
-1
0
2
-2
1
3
3
1
W
1
2
0
-1
w
Output
9
0
1
3
W − w + 1


Input
1
4
-1
0
2
-2
1
3
3
1
W
1
2
0
-1
w
Output
9
0
1
3
-5
W − w + 1


Input
1 4 -1 0 2 -2 1 3 3 1
W
1 2 0 -1
w
Output
9 0 1 3 -5 -3
W − w + 1


Input
1 4 -1 0 2 -2 1 3 3 1
W
1 2 0 -1
w
Output
9 0 1 3 -5 -3 6
W − w + 1


Input
1 4 -1 0 2 -2 1 3 3 1
W
Kernel
1 2 0 -1
w
Output
9 0 1 3 -5 -3 6
$W - w + 1$

Formally, in 1d, given

$$x = (x_1, \dots, x_W)$$

and a "convolution kernel" (or "filter") of width w

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the convolution $x \circledast u$ is a vector of size $W - w + 1$, with

$$\begin{aligned}(x \circledast u)_i &= \sum_{j=1}^{w} x_{i-1+j}\, u_j \\ &= (x_i, \dots, x_{i+w-1}) \cdot u\end{aligned}$$

for instance

$$(1, 2, 3, 4) \circledast (3, 2) = (3 + 4, 6 + 6, 9 + 8) = (7, 12, 17).$$

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⚠ This differs from the usual convolution since the kernel and the signal are both visited in increasing index order.

Convolution can implement in particular differential operators, *e.g.*

$$(0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4) \circledast (-1, 1) = (0, 0, 0, 1, 1, 1, 1, 0, 0, 0).$$

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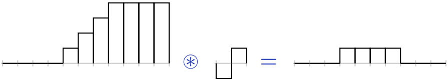

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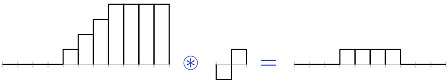
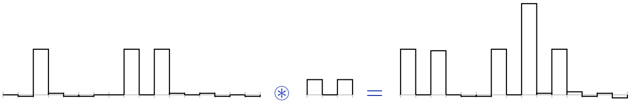
$$(0,0,0,0,1,2,3,4,4,4,4) \circledast (-1,1) = (0,0,0,1,1,1,1,0,0,0).$$



or crude "template matcher", *e.g.*



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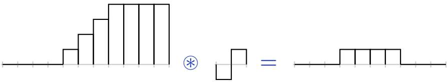

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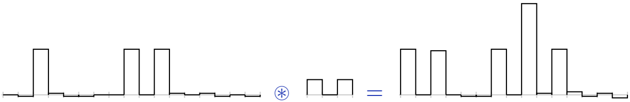

Both of these computation examples are indeed "invariant by translation".

It generalizes naturally to a multi-dimensional input, although specification can become complicated.

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Its most usual form for "convolutional networks" processes a 3d tensor as input (*i.e.* a multi-channel 2d signal) to output a 2d tensor. The kernel is not swiped across channels, just across rows and columns.

In this case, if the input tensor is of size $C \times H \times W$, and the kernel is $C \times h \times w$, the output is $(H - h + 1) \times (W - w + 1)$.

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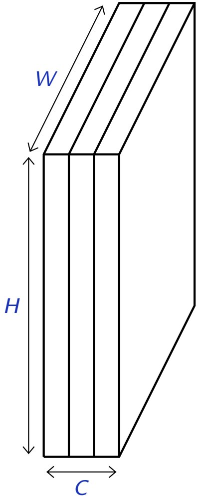
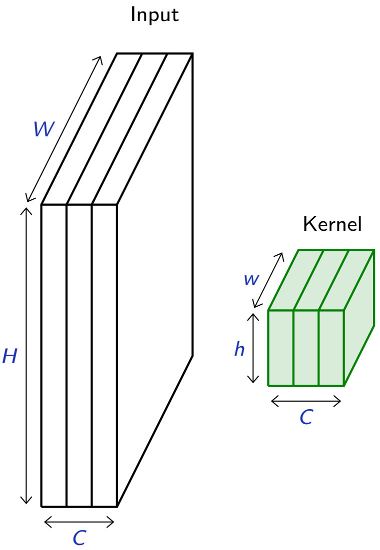
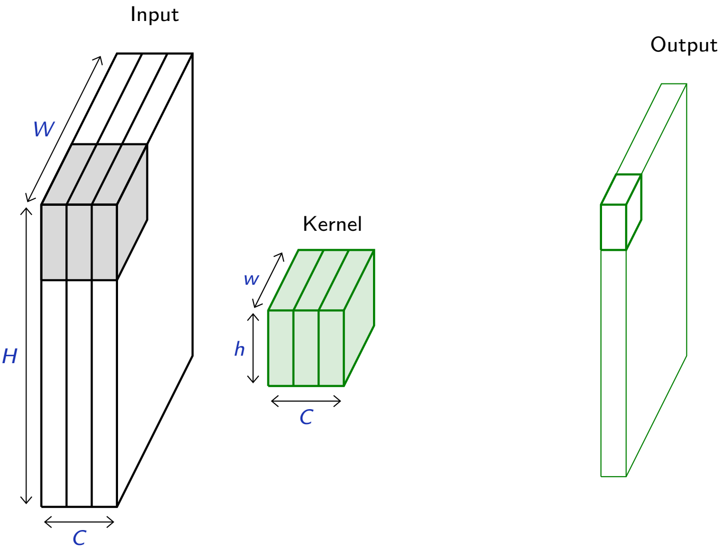
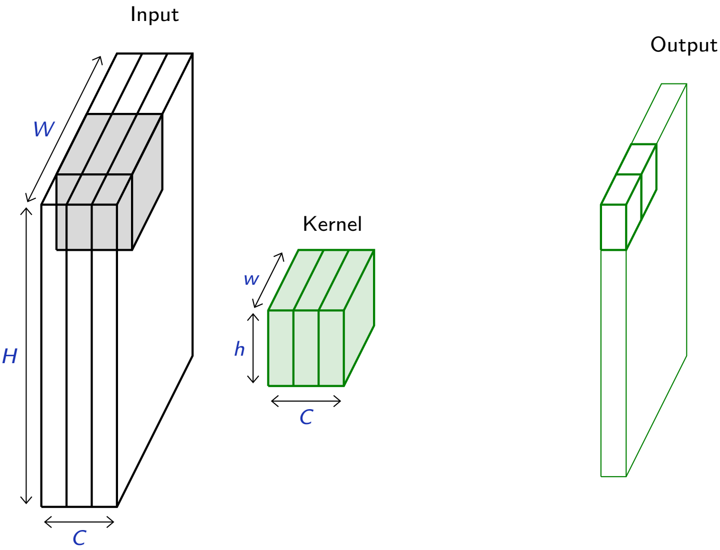
In a standard convolution layer, D such convolutions are combined to generate a $D \times (H - h + 1) \times (W - w + 1)$ output.

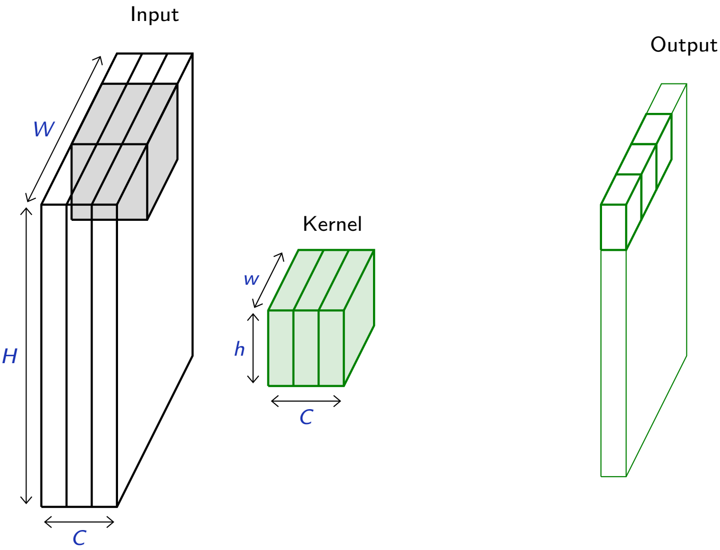
Input

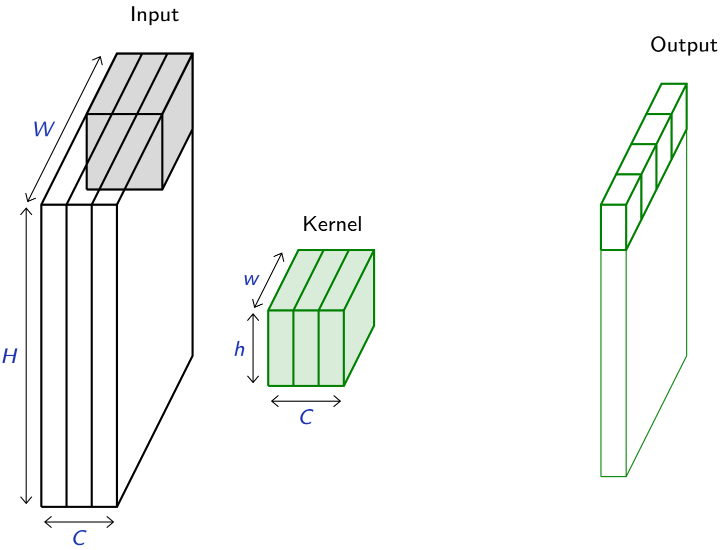


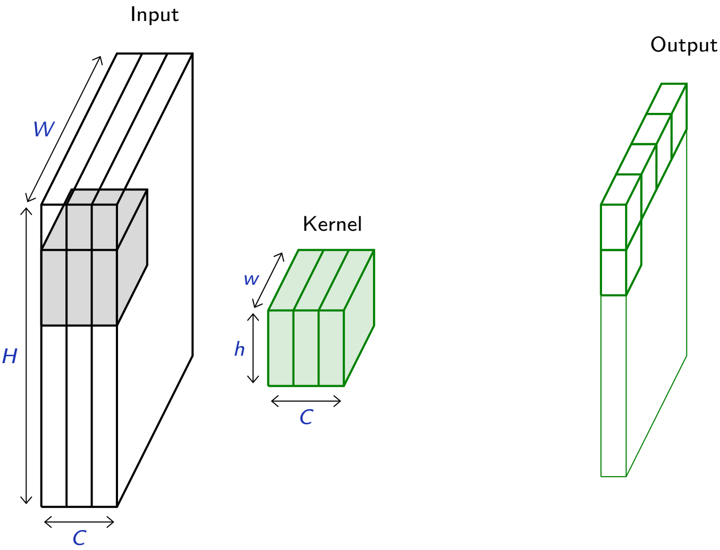

Input
W
H
C
Kernel
w
h
C


Input
W
H
C
Kernel
w
h
C
Output

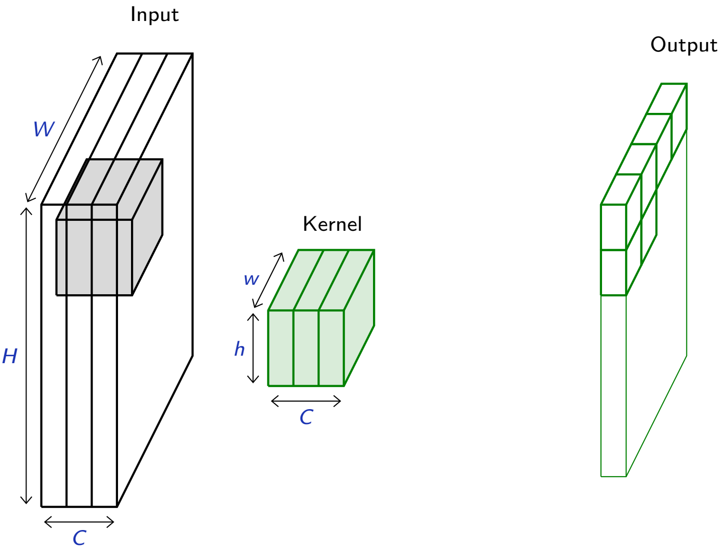

Input
W
H
C
Kernel
w
h
C
Output

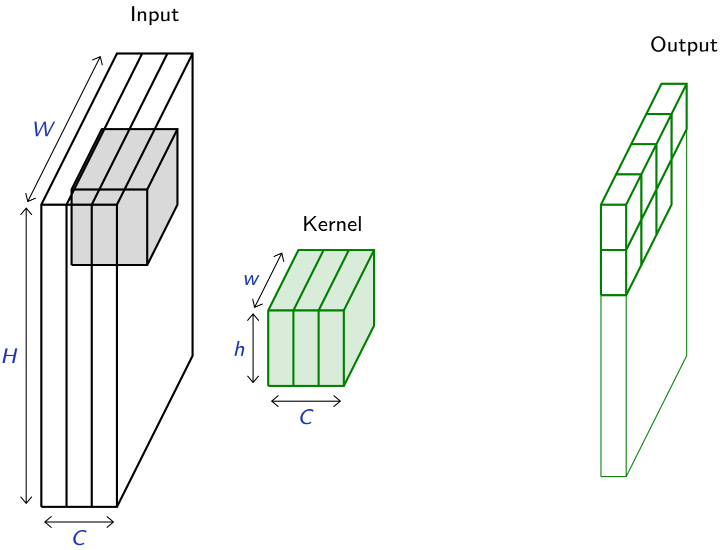

Input
W
H
C
Kernel
w
h
C
Output

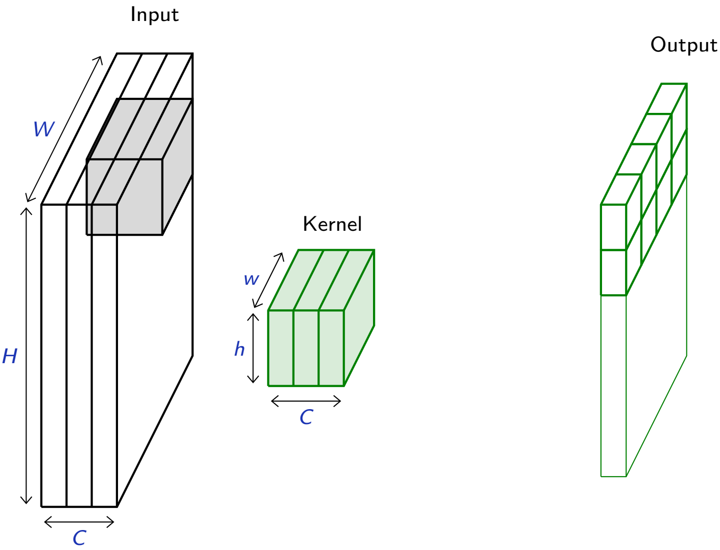

Input
W
H
C
Kernel
w
h
C
Output

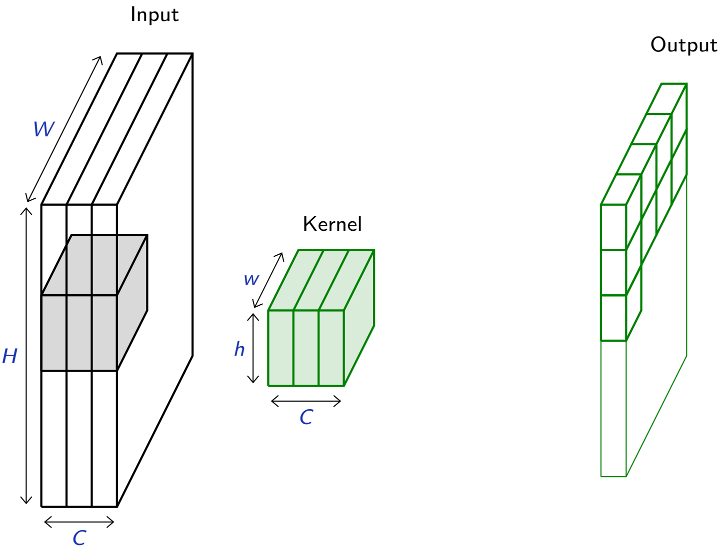


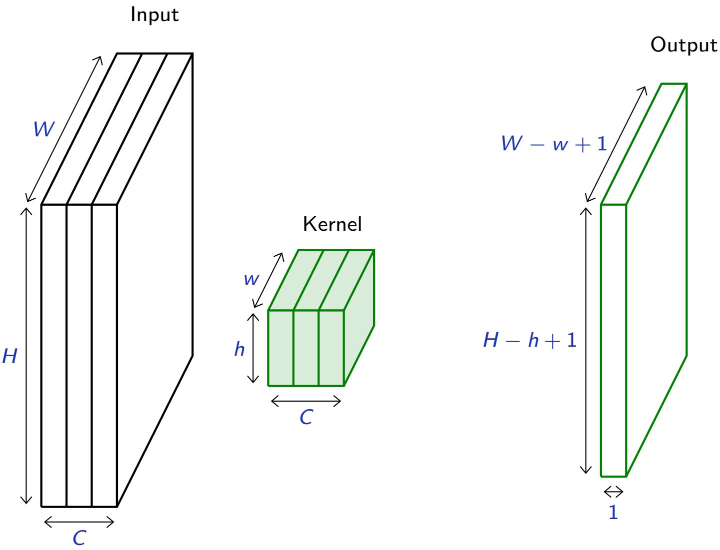
Input
W
H
C
Kernel
w
h
C
Output

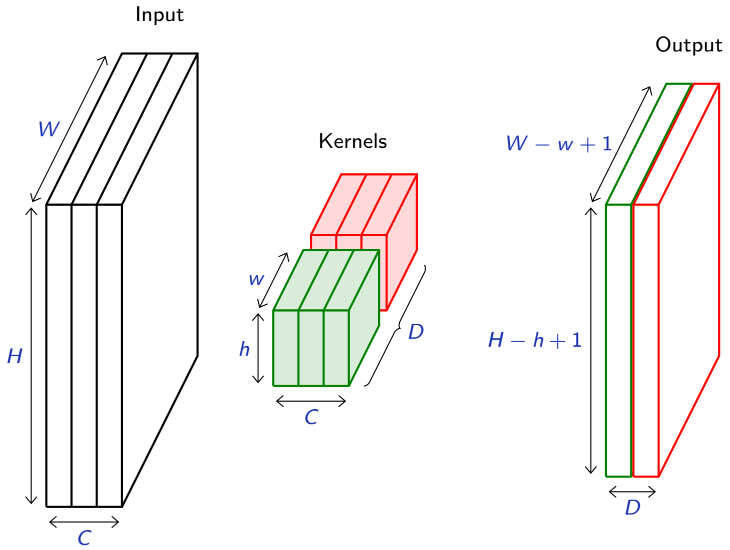

Input
W
H
C
Kernel
w
h
C
Output

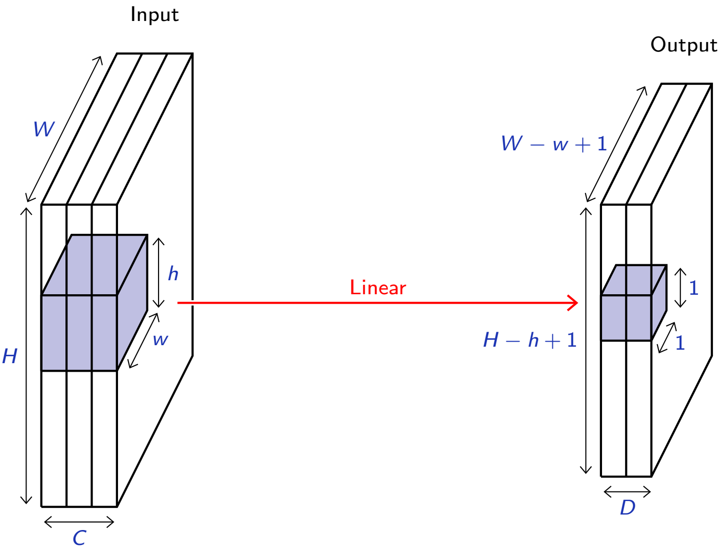

Input
W
H
C
Kernel
w
h
C
Output


Input
W
H
C
Kernel
w
h
C
Output


Input
W
H
C
Kernel
w
h
C
Output


Input
W
H
C
Kernel
w
h
C
Output
W − w + 1
H − h + 1
1


Input
W
H
C
Kernels
w
h
C
D
Output
W − w + 1
H − h + 1
D


Input
Output
W
H
h
w
C
Linear
W − w + 1
H − h + 1
1
1
D

Note that a convolution **preserves the signal support structure**.

A 1d signal is converted into a 1d signal, a 2d signal into a 2d, and neighboring parts of the input signal influence neighboring parts of the output signal.

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A 3d convolution can be used if the channel index has some metric meaning, such as time for a series of grayscale video frames. Otherwise swiping across channels makes no sense.

We usually refer to one of the channels generated by a convolution layer as an **activation map.**

The sub-area of an input map that influences a component of the output as the **receptive field** of the latter.

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In the context of convolutional networks, a standard linear layer is called a **fully connected layer** since every input influences every output.

```
F.conv2d(input, weight, bias=None, stride=1, padding=0, dilation=1, groups=1)
```

Implements a 2d convolution, where `weight` contains the kernels, and is $D \times C \times h \times w$, `bias` is of dimension D, `input` is of dimension

$$N \times C \times H \times W$$

and the result is of dimension

$$N \times D \times (H - h + 1) \times (W - w + 1).$$

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```
>>> weight = torch.empty(5, 4, 2, 3).normal_()
>>> bias = torch.empty(5).normal_()
>>> input = torch.empty(117, 4, 10, 3).normal_()
>>> output = F.conv2d(input, weight, bias)
>>> output.size()
torch.Size([117, 5, 9, 1])
```

```
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```

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torch.Size([117, 5, 9, 1])
```

Similar functions implement 1d and 3d convolutions.

```
x = mnist_train.data[12].float().view(1, 1, 28, 28)

weight = torch.empty(5, 1, 3, 3)

weight[0, 0] = torch.tensor([ [  0.,  0.,  0. ],
                              [  0.,  1.,  0. ],
                              [  0.,  0.,  0. ] ])

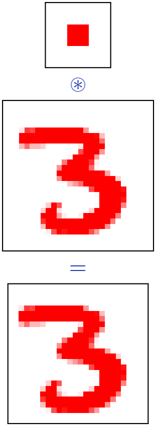
weight[1, 0] = torch.tensor([ [  1.,  1.,  1. ],
                              [  1.,  1.,  1. ],
                              [  1.,  1.,  1. ] ])

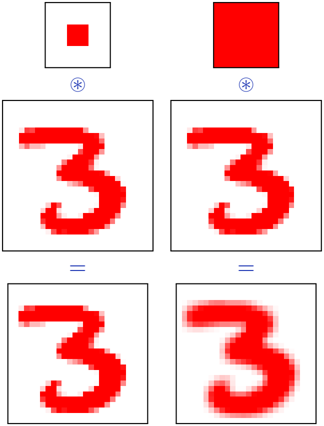
weight[2, 0] = torch.tensor([ [ -1.,  0.,  1. ],
                              [ -1.,  0.,  1. ],
                              [ -1.,  0.,  1. ] ])

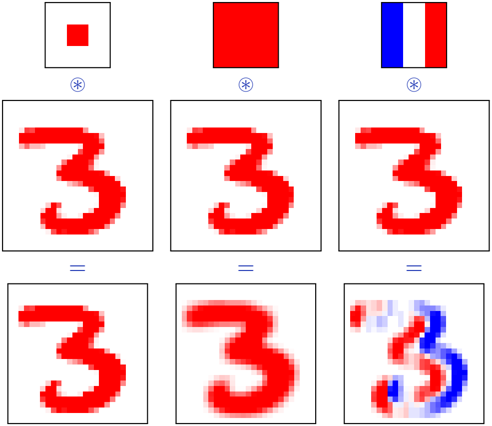
weight[3, 0] = torch.tensor([ [ -1., -1., -1. ],
                              [  0.,  0.,  0. ],
                              [  1.,  1.,  1. ] ])

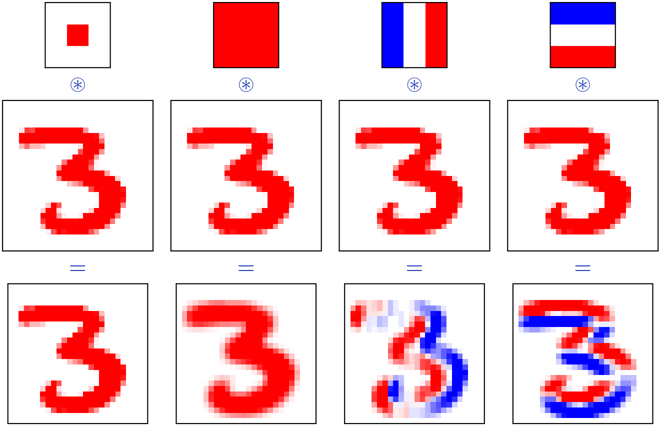
weight[4, 0] = torch.tensor([ [  0., -1.,  0. ],
                              [ -1.,  4., -1. ],
                              [  0., -1.,  0. ] ])

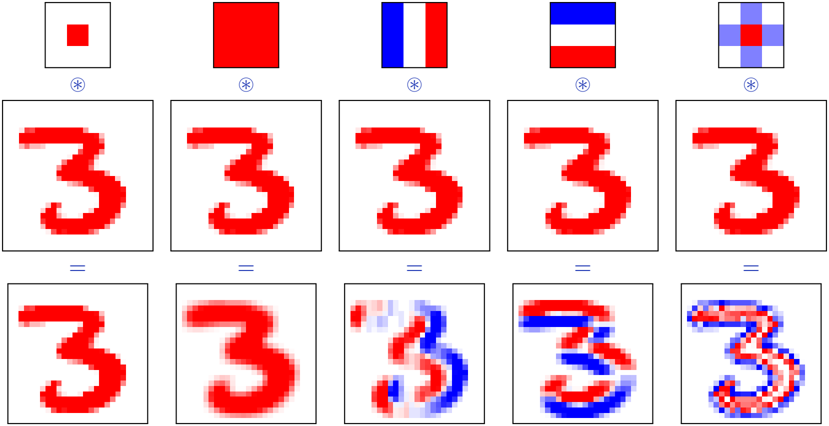
y = F.conv2d(x, weight)
```











```
class torch.nn.Conv2d(in_channels, out_channels,
                      kernel_size, stride=1, padding=0, dilation=1,
                      groups=1, bias=True)
```

Wraps the convolution into a `Module`, with the kernels and biases as `Parameter` properly randomized at creation.

The kernel size is either a pair (h, w) or a single value k interpreted as (k, k).

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```

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The kernel size is either a pair (h, w) or a single value k interpreted as (k, k).

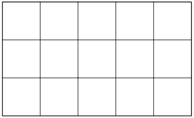
```
>>> f = nn.Conv2d(in_channels = 4, out_channels = 5, kernel_size = (2, 3))
>>> for n, p in f.named_parameters(): print(n, p.size())
...
weight torch.Size([5, 4, 2, 3])
bias torch.Size([5])
>>> x = torch.empty(117, 4, 10, 3).normal_()
>>> y = f(x)
>>> y.size()
torch.Size([117, 5, 9, 1])
```

Padding, stride, and dilation

Convolutions have three additional parameters:

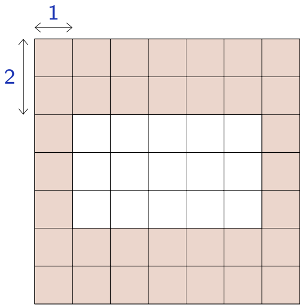
- The **padding** specifies the size of a zeroed frame added around the input,
- the **stride** specifies a step size when moving the kernel across the signal,
- the **dilation** modulates the expansion of the filter without adding weights.

Here with $C \times 3 \times 5$ as input



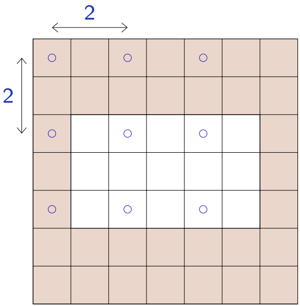
Input

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$



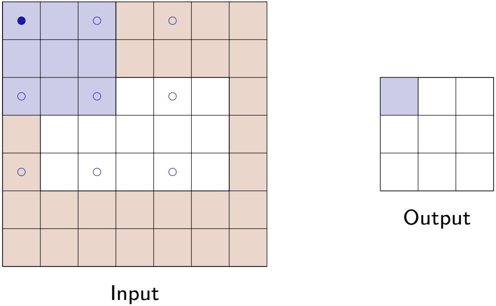
Input

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$



Input

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$



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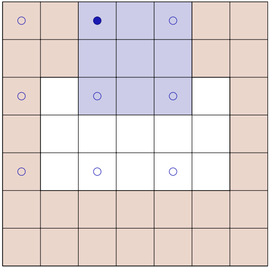
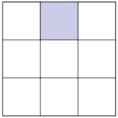


Input



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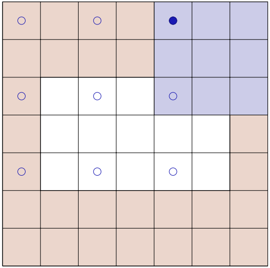
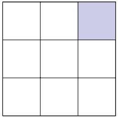


Input



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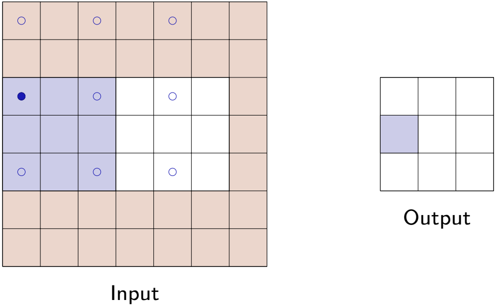

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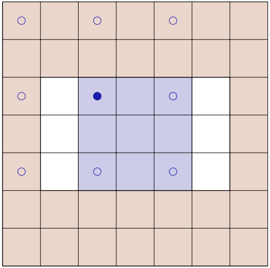
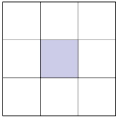


Input



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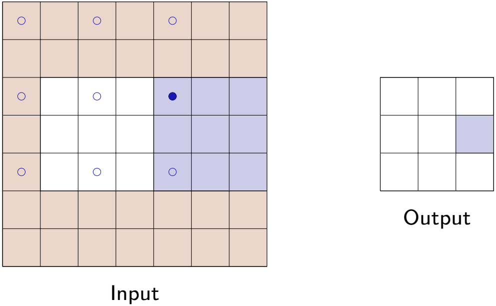

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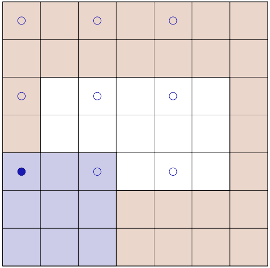
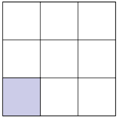


Input



Output

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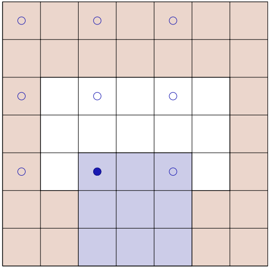
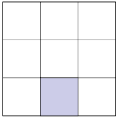
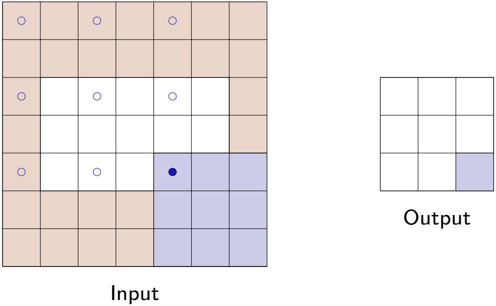


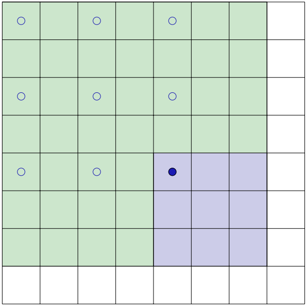
Input



Output

Here with $C \times 3 \times 5$ as input, a padding of $(2, 1)$, a stride of $(2, 2)$, and a kernel of size $C \times 3 \times 3$, the output is $1 \times 3 \times 3$.





A convolution with a stride greater than 1 may not cover the input map completely, hence may ignore some of the input values.

The dilation modulates the expansion of the filter support by adding rows and columns of zeros between coefficients (Yu and Koltun, 2015).

It is 1 for standard convolutions, but can be greater, in which case the resulting operation can be envisioned as a convolution with a regularly sparsified filter.

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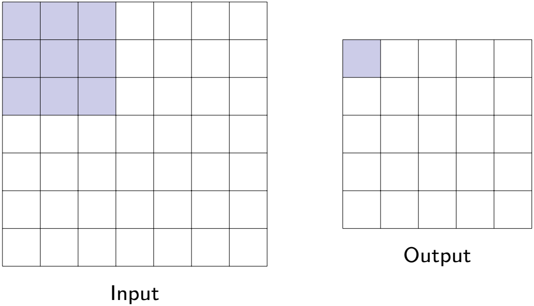
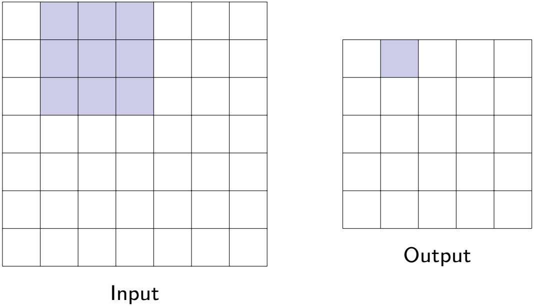
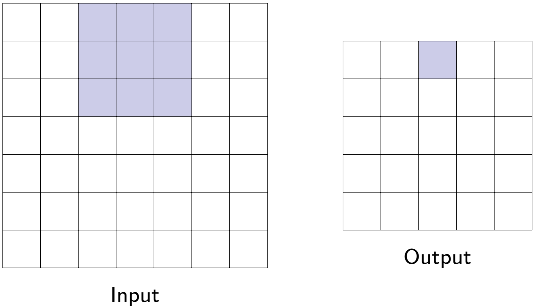
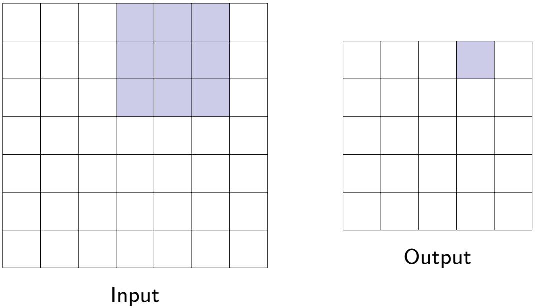
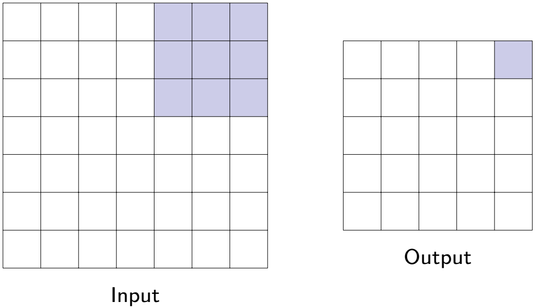
This notion comes from signal processing, where it is referred to as *algorithme à trous*, hence the term sometime used of “convolution à trous”.

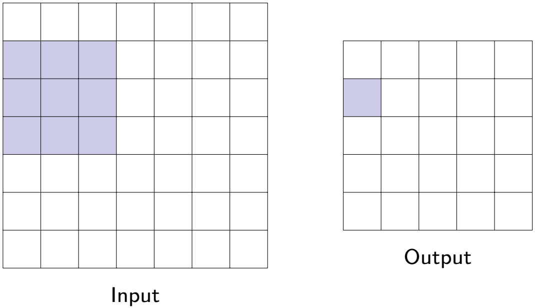

Input
Output

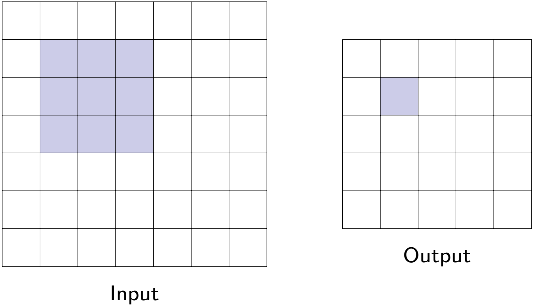

Input
Output

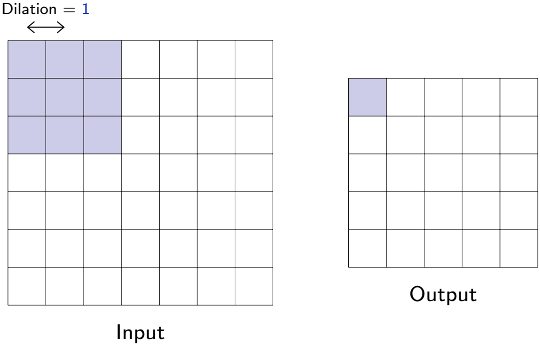

Input
Output

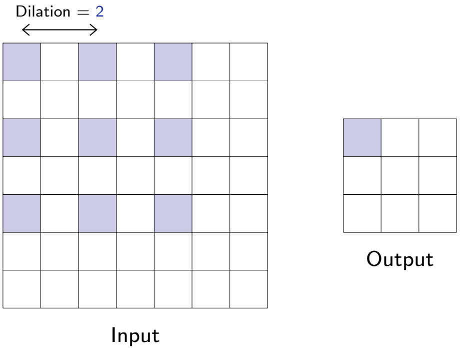

Input
Output

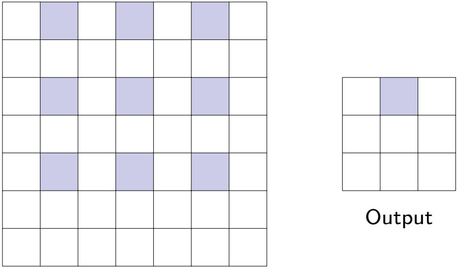

Input
Output


Input
Output


Input
Output

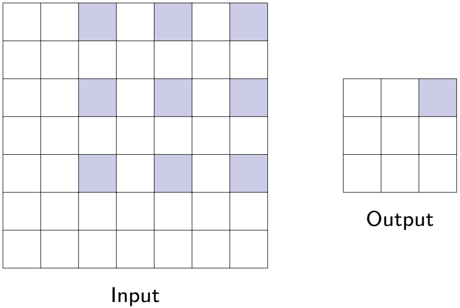

Dilation = 1
Input
Output

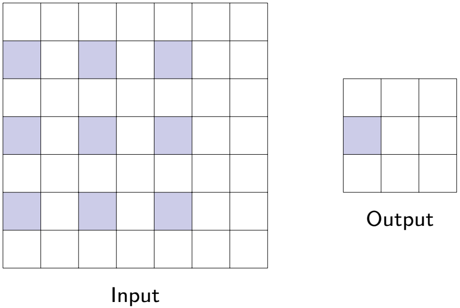

Dilation = 2
Input
Output

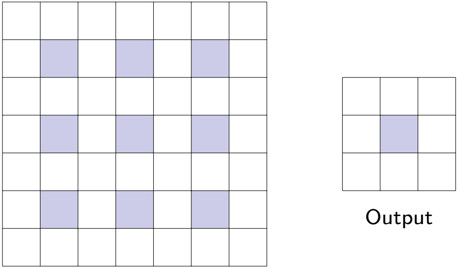


Output

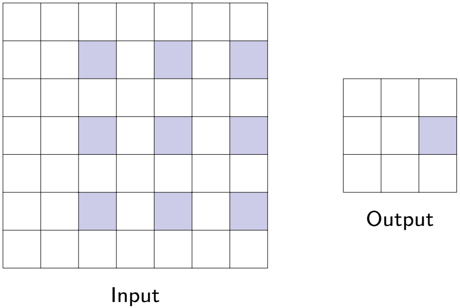
Input

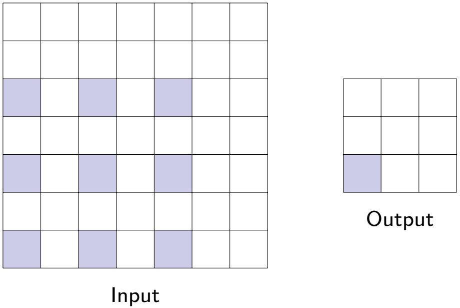

Output
Input

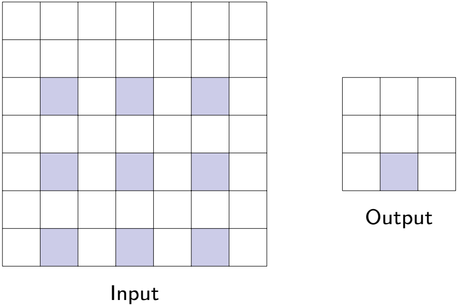

Input
Output

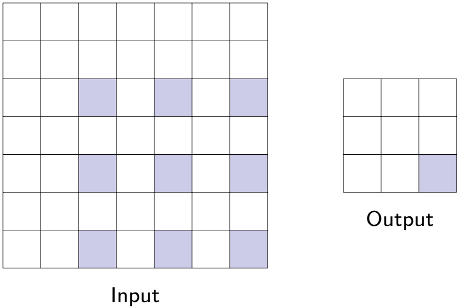


Input


Input
Output


Output
Input


Output
Input


Output
Input

A convolution with a kernel of size k and dilation d can be interpreted as a convolution with a filter of size $1 + (k - 1)d$ with only k non-zero coefficients.

For example with $k = 3$ and $d = 4$, the difference between the input map size and the output map size is $1 + (3 - 1)4 - 1 = 8$.

```
>>> x = torch.empty(1, 1, 20, 30).normal_()
>>> l = nn.Conv2d(1, 1, kernel_size = 3, dilation = 4)
>>> l(x).size()
torch.Size([1, 1, 12, 22])
```

Having a dilation greater than one increases the units' receptive field size without increasing the number of parameters.

Convolutions with stride or dilation strictly greater than one reduce the activation map size, for instance to make a final classification decision.

The end

References

F. Yu and V. Koltun. **Multi-scale context aggregation by dilated convolutions**. CoRR, abs/1511.07122v3, 2015.