

EE-559 – Deep learning

2.1. Loss and risk

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Feb 26, 2020

The general objective of machine learning is to capture regularity in data to make predictions.

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There are multiple types of inference that we can roughly split into three categories:

- Classification (e.g. object recognition, cancer detection, speech processing),
- regression (e.g. customer satisfaction, stock prediction, epidemiology), and
- density estimation (e.g. outlier detection, data visualization, sampling/synthesis).

The standard formalization considers a measure of probability

$$\mu_{X,Y}$$

over the observation/value of interest, and i.i.d. training samples

$$(x_n, y_n), \quad n = 1, \dots, N.$$

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So the quantity

$$\mu_{X|Y=y}$$

stands for the distribution of the observable signal for the class y (e.g. “sound of an /ē/”, “image of a cat”).

For regression, one would interpret the joint law more naturally as

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In the simple cases

$$Y = f(X) + \epsilon$$

where f is the deterministic dependency between x and y , and ϵ is a random noise, independent of X .

With such a probabilistic perspective, we can more precisely define the three types of inferences we introduced before:

Classification,

- (X, Y) random variables on $\mathcal{X} = \mathbb{R}^D \times \{1, \dots, C\}$,
- we want to estimate $\operatorname{argmax}_y P(Y = y \mid X = x)$.

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Density estimation,

- X random variable on $\mathcal{X} = \mathbb{R}^D$,
- we want to estimate μ_X .

The boundaries between these categories are fuzzy:

- Regression allows to do classification through class scores.
- Density models allow to do classification thanks to Bayes' law.

etc.

Risk, empirical risk

Learning consists of finding in a set \mathcal{F} of functionals a “good” f^* (or its parameters’ values) usually defined through a loss

$$\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}$$

such that $\ell(f, z)$ increases with how wrong f is on z .

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- for classification:

$$\ell(f, (x, y)) = \mathbf{1}_{\{f(x) \neq y\}},$$

- for regression:

$$\ell(f, (x, y)) = (f(x) - y)^2,$$

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$$\ell(q, z) = -\log q(z).$$

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The loss may include additional terms related to f itself.

We are looking for an f with a small **expected risk**

$$R(f) = \mathbb{E}_Z (\ell(f, Z)),$$

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Although this quantity is unknown, if we have i.i.d. training samples

$$\mathcal{D} = \{Z_1, \dots, Z_N\},$$

we can compute an estimate, the **empirical risk**:

$$\hat{R}(f; \mathcal{D}) = \hat{\mathbb{E}}_{\mathcal{D}}(\ell(f, Z)) = \frac{1}{N} \sum_{n=1}^N \ell(f, Z_n).$$

We have

$$\mathbb{E}_{Z_1, \dots, Z_N} \left(\hat{R}(f; \mathcal{D}) \right) = \mathbb{E}_{Z_1, \dots, Z_N} \left(\frac{1}{N} \sum_{n=1}^N \ell(f, Z_n) \right)$$

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The empirical risk is an **unbiased estimator** of the expected risk.

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For instance if $|\mathcal{F}| = 1$, we can!

Note that in practice, we call “loss” both the functional

$$\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}$$

and the empirical risk minimized during training

$$\mathcal{L}(f) = \frac{1}{N} \sum_{n=1}^N \ell(f, z_n).$$

The end