EE-559 – Deep learning

6.1. Benefits of depth

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https://fleuret.org/ee559/
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For image classification for instance, there has been a trend toward deeper architectures to improve performance.

Network	Nb. layers
LeNet5 (leCun et al., 1998)	5
AlexNet (Krizhevsky et al., 2012)	8
VGG (Simonyan and Zisserman, 2014)	11–19
GoogleLeNet (Szegedy et al., 2015)	22
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A theoretical analysis provides an intuition of how a network's output "irregularity" grows linearly with its width and exponentially with its depth.

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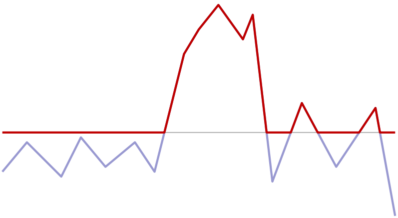

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If we compose σ and $f \in \mathscr{F}$, any linear piece that does not cross 0 remains a single piece or disappears, and one that does cross 0 breaks into two, hence

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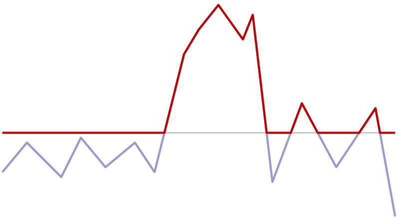

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$$\forall f \in \mathscr{F},\ \kappa(\sigma(f)) \leq 2\kappa(f),$$

and we also have

$$\forall (f,g) \in \mathscr{F}^2,\ \kappa(f+g) \leq \kappa(f) + \kappa(g).$$

Consider a MLP with ReLU, a single input unit, and a single output unit.

$$x_1^0 = x,$$

$$\forall d = 1, \dots, D, \forall i, \quad \begin{cases} s_i^d & = \sum_{j=1}^{W^{d-1}} w_{i,j}^d x_j^{d-1} + b_i^d \\ x_i^d & = \sigma(s_i^d) \end{cases}$$

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$$\forall l, i, \kappa\left(x_i^l\right) = \kappa\left(\sigma(s_i^l)\right) \leq 2\kappa\left(s_i^l\right) \leq 2 \sum_{j=1}^{W_{l-1}} \kappa\left(x_j^{l-1}\right)$$

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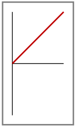
from which

$$\forall l, \max_i \kappa\left(x^l_i\right) \leq 2 W_{l-1} \max_j \kappa\left(x^{l-1}_j\right)$$

and we get the following bound for any ReLU MLP

$$\kappa(y) \leq 2^D \prod_{d=1}^{D} W_d.$$

Although this seems quite a pessimist bound, we can hand-design a network that [almost] reaches it:



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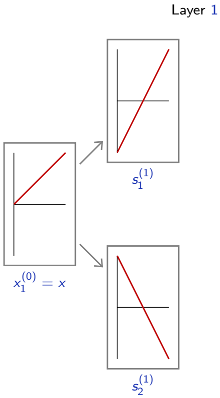

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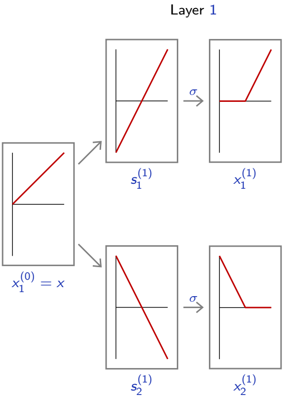

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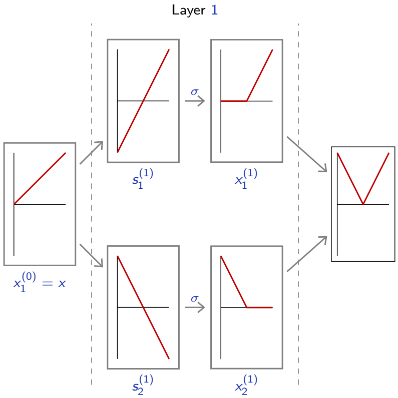

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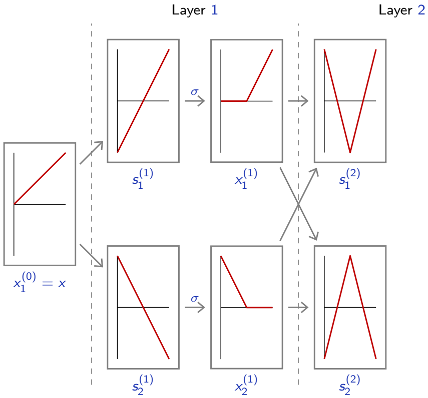

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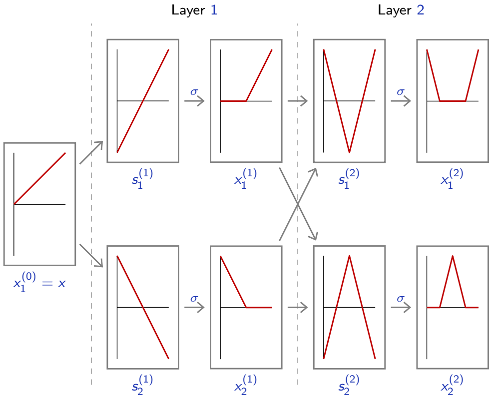

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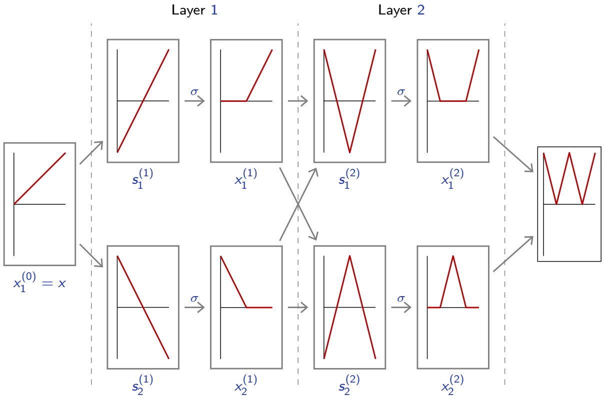

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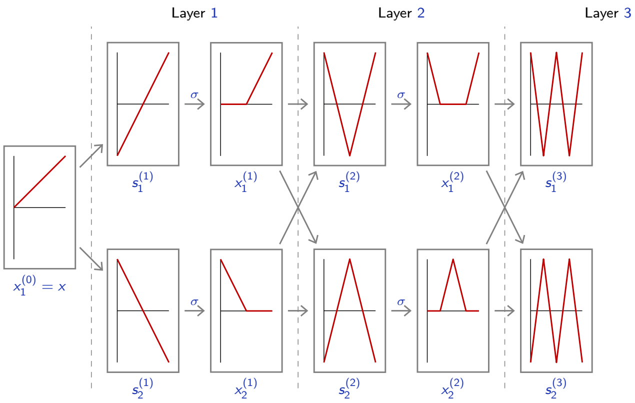

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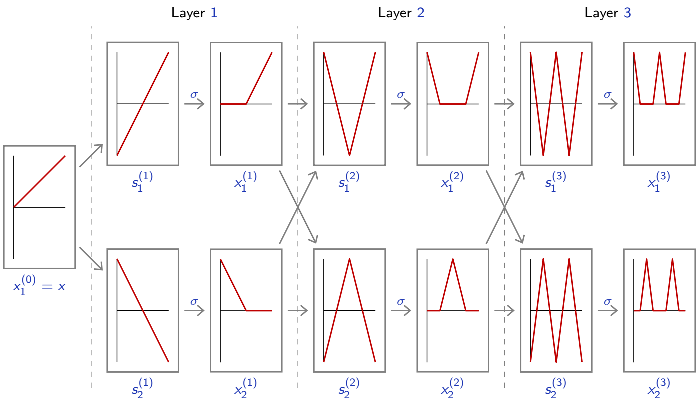

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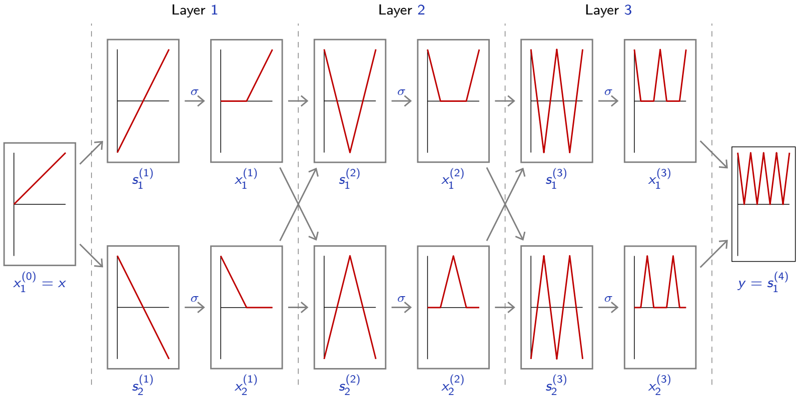
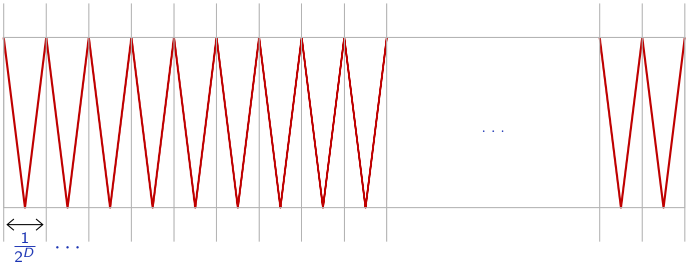
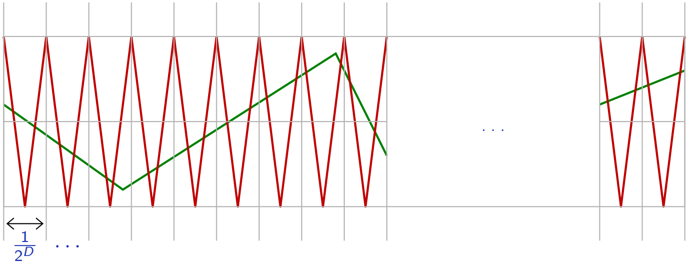


So for any D, there is a network with D hidden layers and $2D$ hidden units which computes an $f : [0, 1] \to [0, 1]$ of period $1/2^D$





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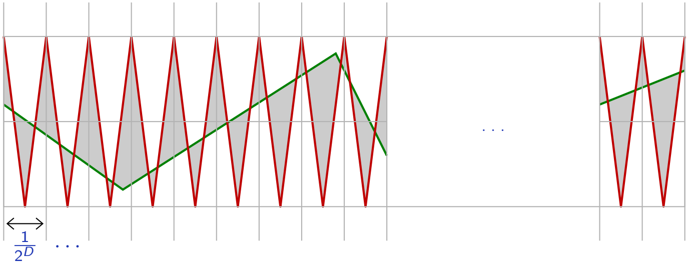

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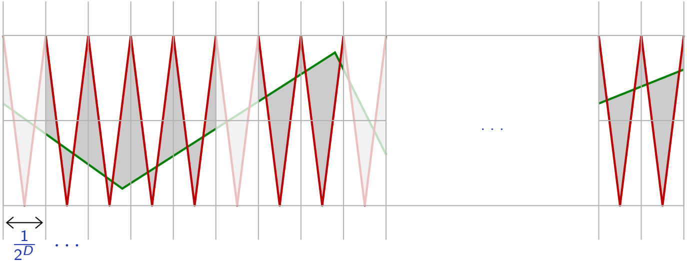
$$\int_0^1 |f(x) - g(x)|$$



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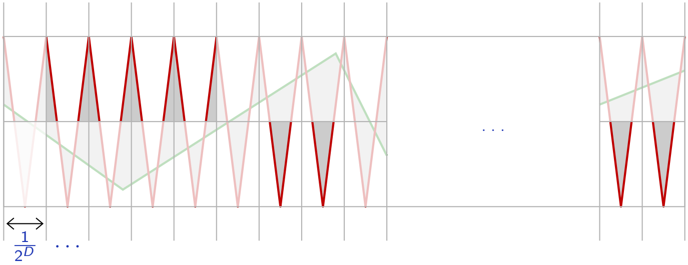

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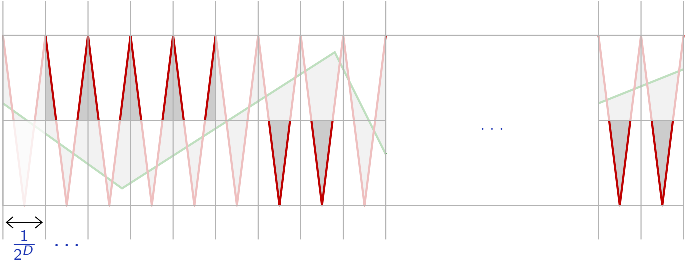

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$$\int_0^1 |f(x) - g(x)| \geq \left(2^D - \kappa(g)\right) \frac{1}{2} \int_0^{1/2^D} \left| f(x) - \frac{1}{2} \right|$$
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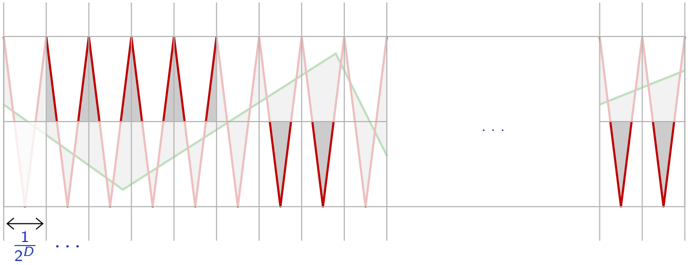

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And we multiply f by 16 to get our final result.

So, considering ReLU MLPs with a single input/output:

There exists a network f with D^* layers, and $2D^*$ internal units, such that, for any network g with D layers of sizes $\{W_1, \dots, W_D\}$:

$$\|f - g\|_1 \geq 1 - \frac{2^D}{2^{D^*}} \prod_{d=1}^{D} W_d.$$

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This is a simplified variant of results by Telgarsky (2015, 2016).

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In particular we have to ensure that

- the gradient does not "vanish" (Bengio et al., 1994; Hochreiter et al., 2001),
- gradient amplitude is homogeneous so that all parts of the network train at the same rate (Glorot and Bengio, 2010),
- the gradient does not vary too unpredictably when the weights change (Balduzzi et al., 2017).

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An additional issue for training very large architectures is the computational cost, which often turns out to be the main practical problem.

The end

References

D. Balduzzi, M. Frean, L. Leary, J. Lewis, K. Wan-Duo Ma, and B. McWilliams. The shattered gradients problem: If resnets are the answer, then what is the question? CoRR, abs/1702.08591, 2017.

Y. Bengio, P. Simard, and P. Frasconi. Learning long-term dependencies with gradient descent is difficult. IEEE Transactions on Neural Networks, 5(2):157–166, Mar. 1994.

X. Glorot and Y. Bengio. Understanding the difficulty of training deep feedforward neural networks. In International Conference on Artificial Intelligence and Statistics (AISTATS), 2010.

K. He, X. Zhang, S. Ren, and J. Sun. Deep residual learning for image recognition. CoRR, abs/1512.03385, 2015.

K. He, X. Zhang, S. Ren, and J. Sun. Identity mappings in deep residual networks. CoRR, abs/1603.05027, 2016.

S. Hochreiter, Y. Bengio, P. Frasconi, and J. Schmidhuber. Gradient Flow in Recurrent Nets: the Difficulty of Learning Long-Term Dependencies, pages 237–243. IEEE Press, 2001.

G. Huang, Y. Sun, Z. Liu, D. Sedra, and K. Q. Weinberger. Deep networks with stochastic depth. CoRR, abs/1603.09382, 2016.

A. Krizhevsky, I. Sutskever, and G. Hinton. Imagenet classification with deep convolutional neural networks. In Neural Information Processing Systems (NIPS), 2012.

Y. LeCun, B. Boser, J. S. Denker, D. Henderson, R. E. Howard, W. Hubbard, and L. D. Jackel. Backpropagation applied to handwritten zip code recognition. Neural Computation, 1(4):541–551, 1989.

Y. leCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.

K. Simonyan and A. Zisserman. Very deep convolutional networks for large-scale image recognition. CoRR, abs/1409.1556, 2014.

C. Szegedy, W. Liu, Y. Jia, P. Sermanet, S. Reed, D. Anguelov, D. Erhan, V. Vanhoucke, and A. Rabinovich. Going deeper with convolutions. In Conference on Computer Vision and Pattern Recognition (CVPR), 2015.

C. Szegedy, S. Ioffe, and V. Vanhoucke. Inception-v4, inception-resnet and the impact of residual connections on learning. CoRR, abs/1602.07261, 2016.

M. Telgarsky. Representation benefits of deep feedforward networks. CoRR, abs/1509.08101, 2015.

M. Telgarsky. Benefits of depth in neural networks. CoRR, abs/1602.04485, 2016.