

EE-559 – Deep learning

5.4. L_2 and L_1 penalties

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We have motivated the use of a loss with a Bayesian formulation combining the probability of the data given the model and the probability of the model

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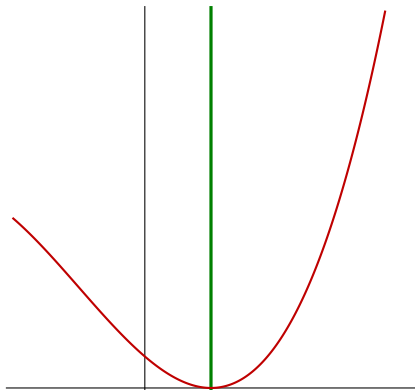
$$\lambda \|w\|_2^2.$$

Since this penalty is convex, its sum with a convex functional is convex.

This is called the L_2 regularization, or “weight decay” in the artificial neural network community.

Increasing the λ parameter moves the optimal closer to 0, and away from the optimal for the loss alone.

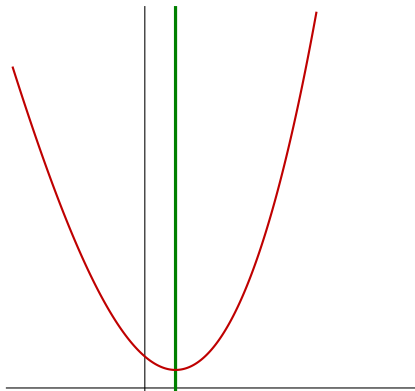
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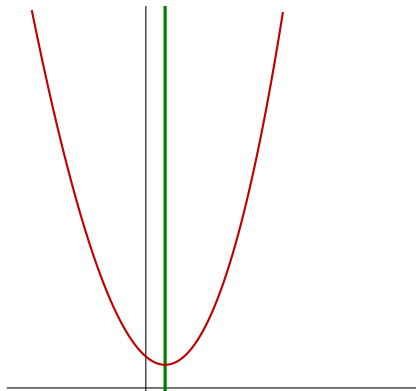
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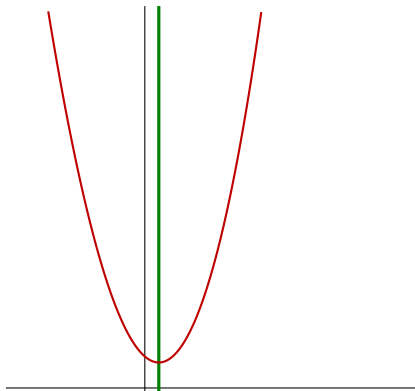
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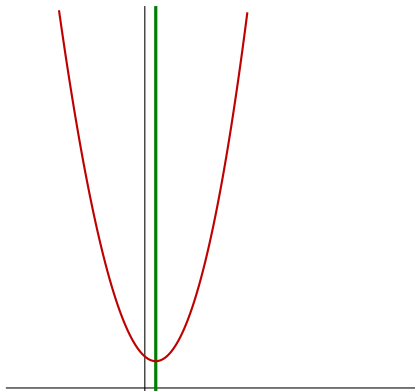
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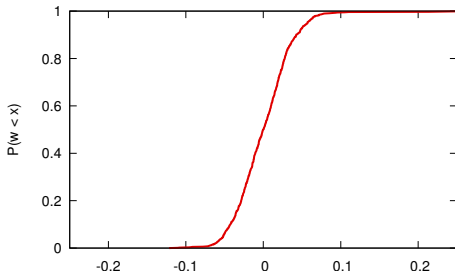
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0.004	0.005	0.065
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output = model(train_input[b:b+batch_size])
loss = criterion(output, train_target[b:b+batch_size])

for p in model.parameters():
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optimizer.zero_grad()
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$\lambda = 0.000$

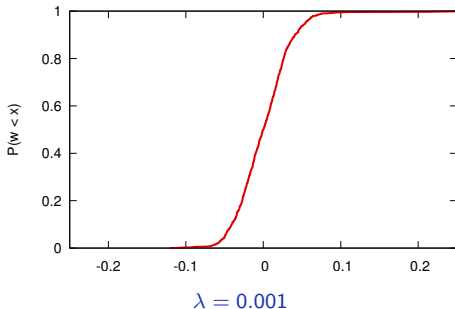
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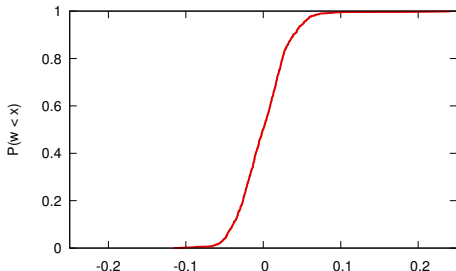
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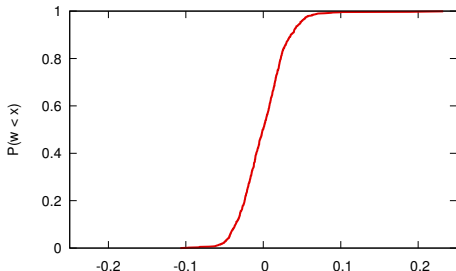
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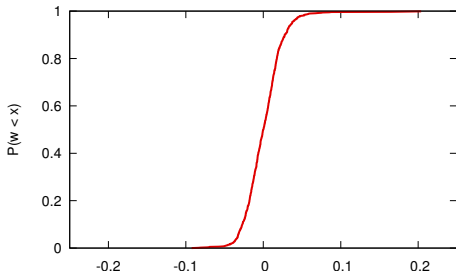
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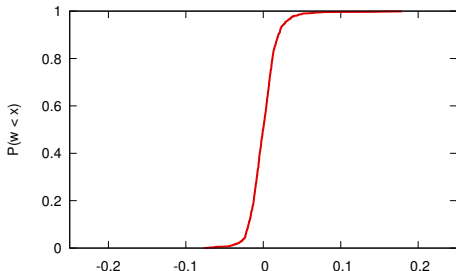
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$\lambda = 0.020$

We can apply the exact same scheme with a Laplace prior

$$\begin{aligned}\mu(w) &= \frac{1}{(2b)^D} \exp\left(-\frac{\|w\|_1}{b}\right) \\ &= \frac{1}{(2b)^D} \exp\left(-\frac{1}{b} \sum_{d=1}^D |w_d|\right),\end{aligned}$$

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which results in a penalty term of the form

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This is the L_1 regularization. As for the L_2 , this penalty is convex, and its sum with a convex functional is convex.

An important property of the L_1 penalty is that, if \mathcal{L} is convex, and

$$w^* = \underset{w}{\operatorname{argmin}} \mathcal{L}(w) + \lambda \|w\|_1$$

then

$$\forall d, \left| \frac{\partial \mathcal{L}}{\partial w_d}(w^*) \right| < \lambda \Rightarrow w_d^* = 0.$$

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In practice it means that this penalty pushes some of the variables to zero, but contrary to the L_2 penalty they actually move and remain there.

The λ parameter controls the sparsity of the solution.

With the L_1 penalty, the update rule becomes

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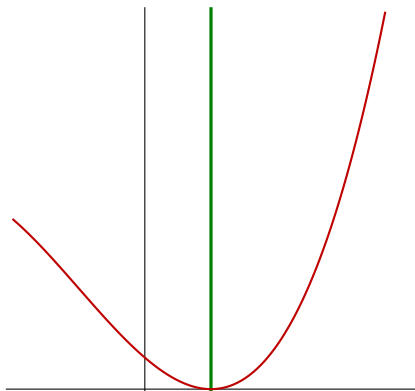
While this is not a problem in principle, since w_t will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).

The **proximal operator** takes care of preventing parameters from “crossing zero”, by adapting λ when it is too large

$$w'_t = w_t - \eta g_t$$
$$w_{t+1} = w'_t - \min(\lambda, |w'_t|) \odot \text{sign}(w'_t).$$

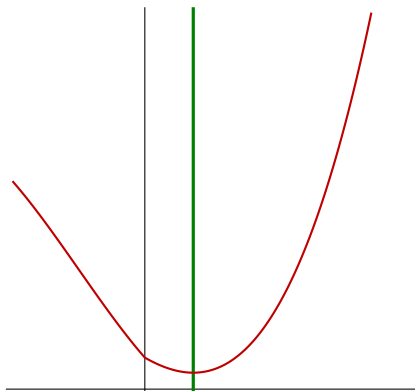
where **min** is component-wise, and \odot is the Hadamard component-wise product.

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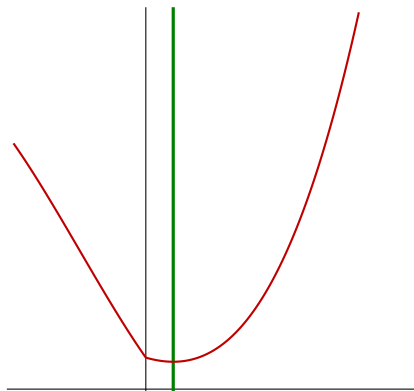
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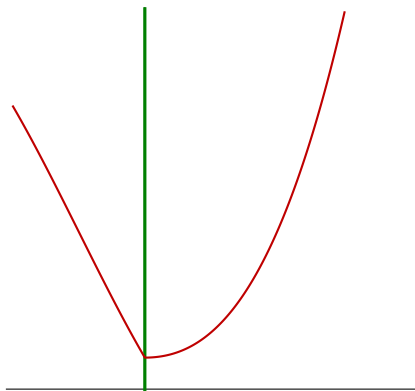
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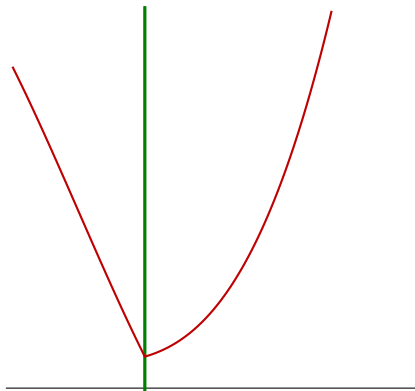
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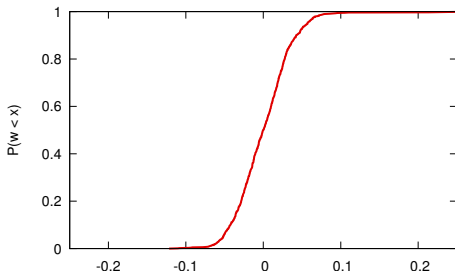
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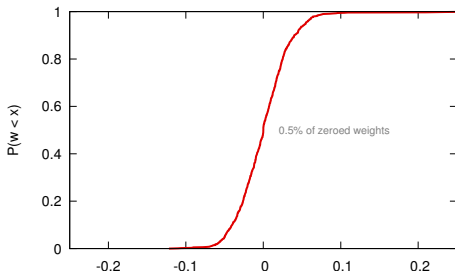
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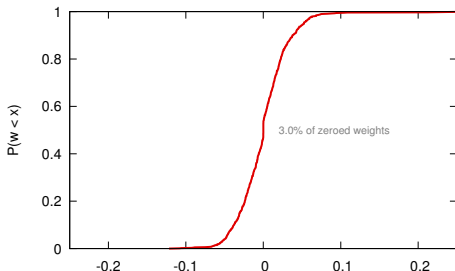
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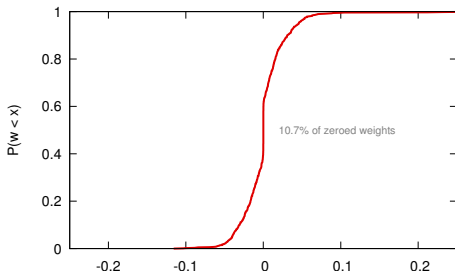
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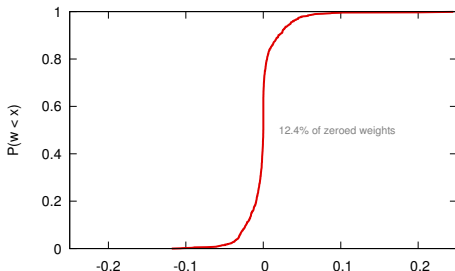
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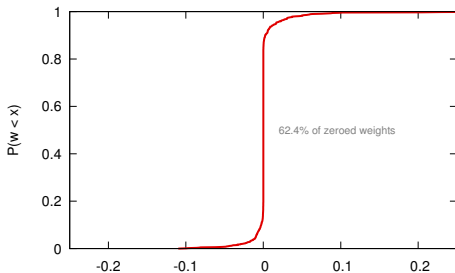
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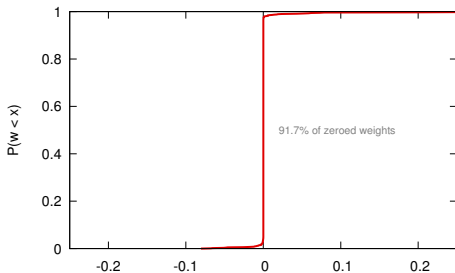
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Penalties on the weights may be useful when dealing with small models and small data-sets and are still standard when data is scarce.

While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines.

The end